DOCUMENT RESUME

ED 356 900 PS 021 385

AUTHOR	Knapp, Nancy F.; Peterson, Penelope L.
TITLE	Understanding Learners' Understandings. Elementary Subjects Center Series No. 97.
INSTITUTION	Michigan State Univ., East Lansing. Inst. for Research on Teaching.
SPONS AGENCY	Office of Educational Research and Improvement (ED), Washington, DC.
PUB DATE	Mar 93
CONTRACT	G0087C0226
NOTE	110p.
AVAILABLE FROM	Center for the Learning and Teaching of Elementary Subjects, Institute for Research on Teaching, 252 Erickson Hall, Michigan State University, East Lansing, MI 48824-1034 ($10.25).
PUB TYPE	Reports - Research/Technical (143)
EDRS PRICE	MF01/PC05 Plus Postage.
DESCRIPTORS	*Elementary School Mathematics; Elementary School Students; Grade 3; *Individual Differences; *Instructional Innovation; *Mathematics Education; Primary Education; *Student Attitudes; Student Behavior; *Teacher Attitudes; Teacher Response; Teaching Methods
IDENTIFIERS	*Discourse Based Teaching

ABSTRACT

This descriptive study explored understandings of five learners in a third-grade mathematics class during a year in which their teacher attempted to change her teaching method. The study also examined the teacher's self-images of teaching while using the new discourse-based approach. Data was obtained from: (1) interviews with the teacher throughout the year; (2) interviews with each student in October and June; (3) small group problem-solving interviews in mid-June; (4) notes from weekly observations during the school year; (5) videotapes of all 20 days of instruction in a multiplication and division unit taught in April and May; (6) copies of students' mathematics notebooks; (7) students' grades on timed tests of number facts; and (8) students' California Test of Basic Skills scores from tests taken in the spring. The five students included an African-American boy, a Chinese boy, a Cameroonian boy, an Iranian-American girl, and a Malaysian girl. A narrative, interpretive presentation of data throughout the year comprises the majority of this document. Analyses revealed how different students had different experiences and constructed different understandings which related to the students' personalities and expectations of their learning in mathematics. Appendices include summaries of all data analyses; selected questions from student interview protocols; and explanations of categories used to code students' multiplication and division strategies. (MM)

****
* Reproductions supplied by EDRS are the best that can be made *
* from the original document. *
****

$10.25

U.S. DEPARTMENT OF EDUCATION
Office of Educational Research and Improvement
EDUCATIONAL RESOURCES INFORMATION CENTER (ERIC)

☒ This document has been reproduced as received from the person or organization originating it.

☐ Minor changes have been made to improve reproduction quality

- Points of view or opinions stated in this document do not necessarily represent official OERI position or policy

ED356900

Elementary Subjects Center
Series No. 97

UNDERSTANDING LEARNERS' UNDERSTANDINGS

Nancy F. Knapp and
Penelope L. Peterson

Center for the
Learning and Teaching
of Elementary Subjects

Institute for
Research on Teaching
College of Education
Michigan State University

PS 021385

MSU is an affirmative action/equal opportunity institution

BEST COPY AVAILABLE

Elementary Subjects Center
Series No. 97

UNDERSTANDING LEARNERS' UNDERSTANDINGS

Nancy F. Knapp and
Penelope L. Peterson

Published by

The Center for the Learning and Teaching of Elementary Subjects
Institute for Research on Teaching
252 Erickson Hall
Michigan State University
East Lansing, Michigan 48824-1034

March 1993

This work is sponsored in part by the Center for the Learning and Teaching of Elementary Subjects, Institute for Research on Teaching, Michigan State University. The Center for the Learning and Teaching of Elementary Subjects is funded primarily by the Office of Educational Research and Improvement, U.S. Department of Education. The opinions expressed in this publication do not necessarily reflect the position, policy, or endorsement of the Office or Department (Cooperative Agreement No. G0087C0226).

Center for the Learning and Teaching of Elementary Subjects

The **Center for the Learning and Teaching of Elementary Subjects** was awarded to Michigan State University in 1987 after a nationwide competition. Funded by the Office of Educational Research and Improvement, U.S. Department of Education, the Elementary Subjects Center is a major project housed in the Institute for Research on Teaching (IRT). The program focuses on conceptual understanding, higher order thinking, and problem solving in elementary school teaching of mathematics, science, social studies, literature, and the arts. Center researchers are identifying exemplary curriculum, instruction, and evaluation practices in the teaching of these school subjects; studying these practices to build new hypotheses about how the effectiveness of elementary schools can be improved; testing these hypotheses through school-based research; and making specific recommendations for the improvement of school policies, instructional materials, assessment procedures, and teaching practices. Research questions include, What content should be taught when teaching these subjects for understanding and use of knowledge? How do teachers concentrate their teaching to use their limited resources best? and In what ways is good teaching subject matter-specific?

The work is designed to unfold in three phases, beginning with literature review and interview studies designed to elicit and synthesize the points of view of various stakeholders (representatives of the underlying academic disciplines, intellectual leaders and organizations concerned with curriculum and instruction in school subjects, classroom teachers, state- and district-level policymakers) concerning ideal curriculum, instruction, and evaluation practices in these five content areas at the elementary level. Phase II involves interview and observation methods designed to describe current practice, and in particular, best practice as observed in the classrooms of teachers believed to be outstanding. Phase II also involves analysis of curricula (both widely used curriculum series and distinctive curricula developed with special emphasis on conceptual understanding and higher order applications), as another approach to gathering information about current practices. In Phase III, models of ideal practice will be developed, based on what has been learned and synthesized from the first two phases, and will be tested through classroom intervention studies.

The findings of Center research are published by the IRT in the Elementary Subjects Center Series. Information about the Center is included in the IRT Communication Quarterly (a newsletter for practitioners) and in lists and catalogs of IRT publications. For more information, to receive a list or catalog, or to be placed on the IRT mailing list to receive the newsletter, please write to the Editor, Institute for Research on Teaching, 252 Erickson Hall, Michigan State University, East Lansing, Michigan 48824-1034.

Co-directors:	Jere E. Brophy and Penelope L. Peterson
Senior Researchers:	Patricia Cianciolo, Sandra Hollingsworth, Wanda May, Richard Prawat, Ralph Putnam, Taffy Raphael, Cheryl Rosaen, Kathleen Roth, Pamela Schram, Suzanne Wilson
Editor:	Sandra Gross
Editorial Assistant:	Tom Bowden

Abstract

Analyses of cases of five learners reveals how, within the same third-grade mathematics class, different students had different experiences and constructed different understandings. These different understandings were related to what students were about as learners as well as what the mathematics was about that the students thought they were supposed to learn in this class in which the teacher was experimenting with a new discourse-based approach to learning mathematics. Two students, Harold and Selvaranee, came to share an understanding of the purposes for discussion as a means of creating mathematical knowledge for themselves and others. Another student, Chang, understood the purposes of discourse as a way for him to tell or transmit what he knew to other students. Unlike Chang, Ataia saw open-ended mathematical discourse as a way to consider multiple ideas, methods, and problem solving strategies that were proposed by her peers. But Ataia frequently picked up multiple, often seemingly contradictory, ideas from the classroom discourse, and this tendency, coupled with her already tentative manner, made Ataia frequently appear to be "confused." Finally, there was Calvin who spent much of his time in a world separate from school, a world of his own--"dreamyland". What it meant to know mathematics in Calvin's authentic world seemed very different from what it meant to know in the world of school mathematics according to Calvin. Calvin continued to see school mathematics as learning to come up with right answers even though his teacher saw mathematics as more than that. Calvin refused to "buy into" class discussion because to him it seemed a waste of time; he thought it would be much quicker just to be told the right answer by the teacher so he could learn it.

UNDERSTANDING LEARNERS' UNDERSTANDINGS

Nancy F. Knapp and Penelope L. Peterson[1]

The Context of Teaching for Understanding

In a sense "teaching for understanding" is a tautology. Most teachers have always hoped their students will understand what they are taught. But teachers differ dramatically in the extent to which they emphasize understanding as a goal of their practice (Peterson, Fennema, Carpenter, & Loef, 1989; Peterson, Putnam, Vredevoogd, & Reineke, 1991), and reformers assert that existing educational practice is not resulting in the kind of knowledge and understandings that students will need to live, work, and learn during their lifetimes (National Commission on Excellence in Education, 1984). Elementary students perform adequately at basic skill levels, yet they show limited expertise on mathematical tasks that require problem-solving skills or higher order thinking (National Assessment of Educational Progress, 1983). Scores on SAT tests and College Boards are falling; colleges, businesses, and the military increasingly have to offer remedial education to bring applicants up to minimal levels of literacy and computation. Nearly 40% of 17-year-olds cannot draw inferences from written material; only one-fifth can write a persuasive essay; and only one-third can solve a mathematics problem requiring several steps (National Assessment of Educational Progress).

Why might learners' understanding be so minimal after 12 years of education? What kind of teaching have American students experienced that has led to their current levels of achievement and how might teaching practice be changed to promote greater thinking and understanding by students?

---

[1] Nancy F. Knapp, a doctoral candidate in educational psychology, is a research assistant with the Center for the Learning and Teaching of Elementary Subjects. Penelope L. Peterson, University Distinguished Professor of educational psychology and teacher education is co-director of the Center. The authors wish to thank Janine Remillard and Jim Reineke, who interveiewed children in Ms. Coleman's class. Finally, we wish to express our appreciation to Ms. Coleman and the students in her third-grad class who allowed us to come and visit their mathematics class for a year, and to look, listen, and learn along with them. Ms. Coleman requested that we not use her real name, so we have honored her request in writing this report.

While there has never been a single model of teaching throughout all public schools in the United States, teaching in most American schools has shared some common features. Knowledge is presented to students as fixed and complete even though scientists and scholars have usually characterized their knowledge as contested and as only "the best we know so far." In traditional elementary and secondary education, students are seldom told about the areas of controversy in a field (May, 1989). They are rarely encouraged to develop their own critical judgments and opinions about the material presented (Roth, 1989). Texts and lectures are presented not as products of individual, fallible authors, working within a scholarly community, but rather as authoritative statements of known truth. Rather than focusing on the development of meaning, relationships, and critical facilities in students, educators have concentrated on getting students to acquire discrete facts and procedures. Teaching has been seen as knowledge transmitting rather than knowledge transforming--telling those facts and procedures to students, whose role has traditionally been to listen, absorb what they are taught, and demonstrate their ability to follow accepted procedures to arrive at correct answers on tests (Cohen, 1988, Schoenfeld, 1992). Students who fail to demonstrate adequate recall and procedural knowledge have been labeled as deficient in either ability or application rather than seen as knowledgeable learners who are struggling to make sense of new ideas in light of what they already know and understand (Resnick, 1989). The development of students' knowledge has been seen as individual, and personal, without consideration of the contexts or situations, including the social and linguistic contexts, in which the knowledge is developed and used (Peterson, in press).

Toward Teaching for Understanding

In contrast to these features of traditional teaching, reform documents such as the National Council for Teachers of Mathematics [NCTM] Standards (1989,1991) offer new visions of mathematics teaching that emphasize understanding and problem solving over rote learning and application of algorithms. Inventing new ways of teaching within their own unique situations and enacting them while participating in the often traditional structures of public schooling inevitably poses a challenge for both teachers and students. In recent case analyses, researchers have described the experiences of teachers as they dealt with the tensions and dilemmas involved in moving toward teaching mathematics for understanding (see, for example, Ball, 1990; Peterson, 1990a; Davis, Maher, & Noddings, 1990; Schifter & Fosnot, 1993; Ball & Renquist, 1993). But only in a few cases have researchers considered the experiences of the students of these teachers as they attempt to teach mathematics for understanding (Cobb, Yackel, & Wood, 1989; Cobb, Wood, Yackel, & McNeal, 1992).

In this report we explore the understandings of five learners in a mathematics class where Keisha Coleman,[2] their third-grade teacher, was attempting to change her classroom practice. We examine how Ms. Coleman saw herself as trying to teach for understanding, we investigate her image(s) of this teaching and how it was played out in her classroom. We particularly focus on Ms. Coleman's students--what and how they understood as they learned about mathematics. We explore how these learners saw themselves as math-doers and the interrelationships among their experiences and their teacher's efforts to teach for understanding, including Ms. Coleman's struggles with the tensions she felt in the process of attempting to teach mathematics in new ways.

Our case analyses of Ms. Coleman and her students are based on a year of data collection during the 1989-90 academic year. These data include interviews with Ms. Coleman throughout the year, interviews with each student in October and again in June, small group problem-solving interviews conducted in mid-June, fieldnotes from weekly observations throughout the year, videotapes of all 20 days of instruction in a multiplication and division unit taught in April and May, copies of students' mathematics notebooks, students' grades on timed tests of number facts ("Mad Minutes"), and their CTBS (California Test of Basic Skills) scores from tests taken that spring. (We provide details on methodology and data collection in Appendix A.) As we began to consider our data, we realized that "the students" were not a coherent whole, upon whom we could look for overall "effects." Rather, students were individuals who each brought different beliefs, knowledge, and discourse styles into the classroom, and thus had different experiences in Ms. Coleman's third-grade mathematics class.

---

[2] Keisha Coleman and all student names are pseudonyms.

The Learners

Our analysis focuses on five students and how they experienced learning mathematics in Ms. Coleman's classroom. Selecting five students for case analyses was not easy; each student had a story that was informative, compelling, and unique. Children in Ms. Coleman's school spoke at least 20 different primary languages, and some attended English as a second language classes, although all regular classroom teachers taught in English. The staff and faculty of the school also represented a variety of ethnic and cultural backgrounds, and Ms. Coleman and the principal were African-American. The five learners that we selected represent this ethnic diversity.[3] But we also chose these learners because their experiences brought up what seem to us to be major issues in this sort of teaching.

Harold

Harold was a new African-American student at the school that year. His mother was attending the nearby state university where, according to Harold, she "sometimes" took math and science classes in addition to her other classes. Most students in this public elementary school were the children of undergraduate and graduate students from the nearby state university, and Harold, like his fellow students, lived in the nearby university-subsidized student housing.

---

[3] Ms. Coleman's class reflected the ethnically and culturally diversity of the school, with 6 Asian (including Indo-Chinese) students, 4 Caucasian students, 4 Black students, and one each South American, Iranian, and Filipino. At least 9 of the 17 students had been born outside the United States, and our data set does not include 2 of these students, who were still barely able to speak English by the end of the year. This is not because we do not recognize important issues that need to be addressed in the considering this type of teaching for students with limited English proficiency, but only because our main data came from individual interviews and class discussions, and these students were very silent during both. Ten students were boys and 7 were girls.

At the beginning of the year, Harold held quite traditional ideas about doing mathematics and being "good" at math. He described the second-grade math class at his former school this way: "When we did our math sheets, . . . we just handed them in in a basket on [the teacher's] desk, and she corrected them." Harold believed that Chang was one of the best math students in this year's class, because in class and on Mad Minutes, "most of the times, he gets it done, and all right and everything." He suggested that Chang had gotten so good because he "does it a lot and practices." When asked how he could tell if an answer was right, Harold replied, "Most of the time, I don't check to see if I got it right or wrong, I just do it." As he talked about math during the interview, he kept nervously ducking his head down into his collar, sometimes actually pulling his shirt up over his head. He said that math had been his least favorite subject last year, "because sometimes it's sort of hard, when we do multiplication and that stuff," but added "sometimes it's fun when the teacher times me, because it's like a challenge or something to see if you can finish the page like in a couple of seconds."

On the October interview, Harold demonstrated reasonable facility in the traditional, procedurally oriented type of mathematics reflected in the comments reported above. He was able to solve four of the six addition and subtraction problems we posed[4], mostly by using algorithmic strategies either mentally or on paper. He got an answer of 44 for "50 - 14" because he thought that "zero minus four is four because you're not taking anything away from it, and five take away one is four." Although

---

[4] The problems posed were as follows:
What is 25 + 10?
What number is ten less than 40?
What is 326 - 100?
What is 326-99?

$$\begin{array}{r} 46 \\ +28 \\ \hline \end{array} \qquad \begin{array}{r} 50 \\ -14 \\ \hline \end{array}$$

Harold appeared satisfied with this answer, at the interviewer's suggestion he then reworked the problem using popsicle sticks as manipulatives, starting with 50 and then taking away 14, and getting an answer of 36. In the end, though, Harold decided to stay with his original answer of 44, rather than revise his thinking. He seemed to trust his paper-and-pencil calculation more than the result he had obtained with popsicle sticks. On the word problems, however, Harold used popsicle sticks correctly to solve a problem in which he was asked how to figure out how many packs of markers a teacher would need to buy for her class of 26 children if each group of 4 children was to have a pack of markers. He used a "counting-by" strategy to correctly solve another word problem, dividing 8 sandwiches equally among 4 children. In solving the word problems, Harold may have relied more on his own sense making abilities because he had not yet been "taught" a paper-and-pencil algorithm for division, so he felt he had to work from his own understanding.

Chang

A fellow student of Harold's, Chang, had already been taught the long division algorithm, not by his teacher, but by his parents. Chang was a Chinese boy who had been attending the school for two years before entering Ms. Coleman's third-grade class. Chang seemed to understand and speak English easily, although with occasionally awkward and stilted phraseology. His father held a computer-related job, and Chang frequently worked on mathematics and other subjects at home with his parents, often learning computational procedures well ahead of his peers in school. In his October interview, Chang told us about how he did the long division algorithm at home:

> We don't do divides in this school, but I do divides in my house. Sometimes it gets too easy for me. Sometimes it doesn't. Dividing you have to do a lot of work. See, if you do dividing, it's like one problem divided by another problem. You have to figure out which number, you times the number, you have to minus the number, times it again, and then you have to do it a lot, a lot of times. It takes a lot of time.

Here Chang verbalized the several steps in the long-division procedure that he had been taught. "Division" for Chang meant implementing these procedural steps. Chang also spoke of working at home with his friend, Ali, and in doing so, he told how he liked to practice mathematical calculations and how he valued speed of calculation :

> Like yesterday, I went to Ali's house, . . . we just played something and finally we got so bored at playing something, sometimes, you know, you get bored at playing something. And then we just worked on our times tables and then I said, "Let's do some divides." And he said, "Okay." And then I wrote some divides down and then he did them. Then it was my turn, and I divided them faster. We timed each other.

Chang used his algorithms to good advantage, performing well on the addition and subtraction problems on the initial interview and getting five out of six number problems correct. Chang got the correct answer to 50 - 14 using the traditional algorithm. Yet when asked by the interviewer, Chang could not demonstrate the problem using popsicle sticks as manipulatives, saying finally that, "without the numbers I can't do them [the problems] well." Chang did 25 + 10 in his head. Then when he was asked to do 326 - 99 mentally, he got 215 and described doing the algorithm in his head by visualizing it and "borrowing." Chang solved neither of the two word problems correctly. When the interviewer asked him to draw the sandwiches in the word problem to help him solve it, Chang was unable do so, but wrote numbers instead and still could not solve the problem.

Chang had a lot of self-confidence in math. He said that mathematics had been his favorite subject in second grade, "so I could do very good at [it]," and tests were the best part of math, "because tests are like, your teacher gives you problems and you get to answer them. That's pretty fun for me." In this year's class, Chang listed three kids who he felt were "really good" at math, "Yan, which is my friend, and Ali; I think that's all, including me." In order to get good at math, Chang advocated lots of practice: "You should ask your mom to give you lots of math for homework. You should do 10 [problems] a night, or when you're free. I did 30 before." In order to tell if they had the right answer to a problem, Chang said that he and Ali would

> sometimes use a piece of paper and then write down the correct . . . um, write down the problem and try to figure out fast at the start so other

> people can't get past us. But most of the times we don't have to get a piece of paper, we just figure it out at the same . . . we just figure it out.

Many of Ms. Coleman's other students shared Chang's good opinion of himself; in October he was named by seven fellow classmates, more often than any other student, as someone who was "really good" at math. Ms. Coleman also recognized Chang's mathematical abilities. In October, she spontaneously mentioned Chang as the only one of her students whose mathematics abilities she thought might be as "strong" as those of two students whose mathematical thinking had impressed her when she observed the other third-grade mathematics class in the school that was taught by her colleague, Deborah Ball, and in several other interviews she referred to Chang as one of the students who best understood various mathematic principles.

Calvin

Calvin was one of the classmates who greatly admired Chang's mathematical abilities. Like Chang, Calvin had been attending this school since first grade. During that time, his mother had been studying about "how to make plants grow" at the nearby state university. A black student who was a native of Cameroon, Calvin spoke English fluently and used colloquialisms easily. In the interview at the beginning of the year, Calvin, like most of Ms. Coleman's students, expressed fairly traditional beliefs about mathematical knowledge. Calvin thought Chang was probably really good at math because Chang greeted every math question with cries of "Eeeeasy!" and he figured that Chang had gotten so good at math by practicing at home. During the fall interview, Calvin said that to check a solution when he was working independently, he would use a number chart or his fingers to count on. When students disagreed in class, Calvin said he would "just wait until the answer is given . . . sometimes [by] Chang." Calvin expressed a reasonable self-confidence in math. He said that he didn't mind having his answers challenged in class; he just thought "It's fair." Although art was Calvin's favorite subject, he stated that he liked science and math "sometimes." In fact, Calvin felt that math had been his best subject last year, because "Mrs. Francis gave us some math, and I got everything right."

The interviewer noted her impression that Calvin was an "unusually active, curious, and verbal child, asking numerous questions about the purposes of the interview, how the recording equipment worked, and other topics." Yet Calvin's mathematical performance was only about average for students in this class. Calvin gave correct answers to three of the six computation problems and one of the two word problems. Calvin solved the computation problems using standard algorithms, and he often seemed to be following the procedure without a clear understanding of why the algorithm worked. For example, he got 84 by using the standard carrying algorithm to solve 46 + 28, explaining that he "carried" because otherwise "you'd get a number much too big" (i.e., 814). When Calvin was asked to solve the same problem with popsicle sticks, he got 74 and, while he admitted that the answers were different, he thought both answers were "OK."

Ataia

Ataia was a girl in Ms. Coleman's class whose mathematical performance during the initial interview was similar to Calvin's. She too answered correctly three of the six addition and subtraction problems, and she solved correctly one of the two word problems that were posed. In attempting to solve the problems, Ataia seemed to think algorithmically. For example, she attempted to do the traditional borrowing algorithm to solve 326 - 99, and got very confused in the tens column when she subtracted 2 from 9 to get 7. She finally got an answer of 676 and seemed satisfied with it. She got 44 as an answer to 50 - 14, and when asked to solve this problem with popsicle sticks, attempted unsuccessfully to manipulate the sticks algorithmically by placing the sticks as follows to look like the computation problem:

50 |||||
-14 | |||||

Like many of her peers, Ataia named Chang as a student who was "really good" at math, but she added Arnie to the list as well. She wasn't sure how Chang and Arnie got to be good at math, but she speculated that "maybe they studied it." Ataia was a girl whose father was Iranian and whose mother was from the United States. Along with Chang and Calvin, Ataia had been at the school since first grade, and she spoke English easily and without accent. She had attended kindergarten in Iran when her family lived there. In the fall interview, Ataia said that math was the subject she had done best in last year, "because on my math I always came home with stars and stuff." This year, though, her initial response to math was that she didn't really like it "that much" because it 's "kind of boring, like . . . for our teacher we take times tests, and it's hard"; Ataia recalled a time in class during Mad Minutes when she felt "bad . . . because I was the only one who got up to 20, and the rest got farther than me." When she was doing a math problem, Ataia said she would first write down her answer, and then count on her fingers to see if it was right; then if it were not right, she would "erase it."

Selvaranee

Selvaranee was another girl in Ms. Coleman's class. Although she was from Malaysia, Selvaranee spoke English fluently. This was her first year at this school, having attended school the previous year in Ann Arbor, Michigan. Selvaranee told us her father used mathematics in his work for "writing long numbers." During the October interview, Selvaranee, like most of her peers, evidenced fairly traditional beliefs about mathematical knowledge. She believed that Yan was really good at math because he always got A+ on Mad Minutes, "maybe [because] he studies hard." On the mathematics interview at the beginning of the year, Selvaranee got the same problems correct as Chang. While she solved neither of the two word problems correctly, she solved correctly five of the six addition and subtraction problems, using the traditional algorithms with "borrowing" or "carrying." Unlike Chang, however, Selvaranee could also demonstrate her understanding using manipulatives. For example, she showed how she could also use popsicle sticks to get "74" for 46 + 28. When she was doing a problem, Selvaranee volunteered that she also might "try to count" to see if she had the right answer. She said she liked math, "because it's fun," especially "answering questions and problems," and she felt pretty good about her own abilities. She said "sometimes" people challenged her answers in class, but "it wasn't too bad."

Looking Across Learners' Knowledge and Beliefs at the Beginning of the Year

Ms. Coleman's students began the year with fairly traditional beliefs about learning and knowing in mathematics. When we interviewed them at the beginning of the year 10 out of 13 students indicated that "being good" at math meant getting right answers quickly, and 9 of these believed that students became good at math by practicing in school or at home. The remaining students either were not asked the questions related to these issues or entered the class after we had conducted the initial interviews. When the students were given two word problems to solve related to whole number division (see Appendix B), we found that 10 of 13 students were able to solve the first (which might be mapped onto 8 ÷ 4), while only 3 could solve the second (which might be mapped onto 26 ÷ 4). We did not ask the students to do any purely computational problems in multiplication or division in the initial interview.

From the initial interviews in the fall, we concluded that Harold, Chang, Calvin, Ataia, and Selvaranee had developed certain ways understandings of addition and subtraction; when given addition and subtraction number (computation) problems to solve during the interview situation, the students' responses appeared to be primarily algorithmic and procedural. Just as important as the students' mathematical understandings were their beliefs about mathematical knowledge and about learning mathematics. All five students had well-developed ideas about how one learns mathematics, how a learner knows when a mathematical answer is correct or not, what it feels like to have your answers challenged, and what it means to be good at mathematics. These beliefs became important as lenses through which the learners "saw" what happened in their mathematics class during the rest of the year. For all students, these beliefs influenced the way they came to understand mathematics and what it meant to learn mathematics in Ms. Coleman's classroom, as their teacher experimented with a new way of teaching mathematics. For several students, these beliefs themselves changed in important ways, while other students held rigidly to their traditional ideas about what it meant to be good at mathematics and to know mathematics. In each case, these students' beliefs fit with the interpretations that they had drawn about what they were supposed to be doing and thinking in Ms. Coleman's mathematics class.

The Process of Developing Understanding in Ms. Coleman's Math Class

What happened during Ms. Coleman's mathematics class and how did students understand what was happening? What kinds of mathematics were discussed and how did the classroom discourse ebb and flow? What roles did these students assume in the classroom discussion and what did these students learn from it?

The first day of the multiplication and division unit in Ms. Coleman's math class provided a context for student's subsequent learning during the unit. Both the mathematical tasks and the kind of discourse in which students and teacher were engaged were also typical of the tasks and discourse patterns in Ms. Coleman's third-grade class over the year (see also Peterson, 1992). The beginning of mathematics class on this particular day was not typical because Ms. Coleman began with a "review" by asking students to tell some things they remembered that they did last week. Students recalled that they had been working on doubling and tripling numbers. Chang remembered specifically that Ms. Coleman had asked them to show their calculation of 3 times 8 on the "minicomputer" (a base-2 abacus-like device used in the Comprehensive School Mathematics Program, the adopted curriculum in Ms. Coleman's school). When their teacher asked why they thought they were doubling and tripling numbers last week and "what this was related to," Ali responded "multiplication," and Chang added, "division."

Ms. Coleman affirmed their replies by indicating that she wanted to continue with these concepts today by having the students do some mental calculations for her. She then asked Ali, "What is 4 times 10?" Ali answered, "40," and Ms. Coleman called on him and several other students to explain how they knew the answer was "40." Then Ms. Coleman asked the students about "8 times 10" and "10 times 8"; she wanted to know whether these were "the same or different." Ali said, "the same," and Chang said, "different," but then he proposed a new idea.

Chang's Method Proposed: A New Procedure for Multiplying by Ten

Chang's idea turned out to be a mathematical procedure that he had learned from his mother. Chang proposed that:

> If you times 10 times any number, it'll always [inaudible] adding a number to the end of the number. Take this 0 (pointing to the 0 on the end of the 10) and put it over there (behind the 8). All of the problems, if you times anything to, to 10, even a million times 10, you just have to add another 0 to the end.

In proposing this new "method," Chang was participating in a process that had become fairly routine by then in Ms. Coleman's class. A student would propose a "method," and then discussion would ensue in which students would agree, disagree, or revise this method. When the proposed "method" reached a form in which the class was able to agree on the wording of it and how it would work, Ms. Coleman would write the new "method" on the blackboard and ask students to copy it into their mathematics notebooks. She would also post it the next day on construction paper above the front blackboard, complete with the name of the student who had suggested it. Occasionally, a posted statement was actually more of a general principle, such as the idea that "times is kind of like add and division is kind of like minus" proposed by Yan, but these statements were still referred to as "methods" by Ms. Coleman and her students.

Attempting to Understand What Chang Had Proposed

In this particular instance, after listening attentively to Chang's words, Ms. Coleman turned to the class and asked the students if they understood what Chang was saying. Some students called out, "No!," while others yelled out. "Yes!" Ms. Coleman then turned to each of several other students and asked each of them to explain what Chang was trying to say. Marta went to the board and explained what she thought Chang meant by saying:

> Like this problem (pointing to 4 x 10)--you take this problem, and, every time you have something with a 10, like 4 times 10 or 2 times 10, or whatever, the number that you're adding 10 . . . you add one more 0 so, so it equals the answer.

Ms. Coleman then asked students if they had comments and if they agreed or disagreed with Marta and Chang. Frankie, Melissa, and Arnie each in turn stated that they didn't understand what had been said. But Ataia volunteered that she understood what Marta had said. Ataia went to the board and wrote as she explained:

> Chang said that if you have like, you have 10 times, like 8 (she wrote 10 x 8 = 80 on the board), and it equals 80, all you have to do is add a 0 to it (she wrote a 0 after the factor 8), then you get . . .

Then Ms. Coleman asked her if she could give them another example that was "not up there," but Ataia shook her head. Her teacher continued, "Did you understand what principle, what he was trying to say, in his explanation of multiplying numbers by 10?" When Ataia indicated that she did understand, Ms. Coleman again asked Ataia whether she could think of another number that Chang's method would work for. Ataia replied that she thought it could work for any number. Chang agreed with Ataia's suggestion that it could work for any number and affirmed that:

> Ataia was saying what I was saying. I mean that whenever you have a number times 10, you just have to add the number, that you times by 10, add a 0 to it. Like 10, 10, 100, just add another 0 because it's like 100 ten times. 100 ten times is a thousand.

Ms. Coleman then called on Bert and asked him what he though about that. When Bert said that he didn't know, his teacher admonished him to pay attention.

Thus far this discussion was typical of Ms. Coleman's teaching in that, if a student proposed a useful new way of doing a certain type of problem, Ms. Coleman would lead the student to formulate it verbally. This "method" might be something the student had figured out himself or herself or something shown to the student by parents or others outside the class. She would ask students if they understood the proposed method, what they thought about it. and if they agreed or disagreed with the proposed method. Ms. Coleman tried to avoid telling students directly whether their ideas were right or wrong. Often, rather than replying directly to a student who had spoken, Ms. Coleman would simply turn to another student and ask what that student thought. Ms. Coleman called on students who had their hands raised as well as those who didn't. She seemed to want to get students to verbalize their understanding of what had been said by others, but she also wanted to ensure that students were paying attention. At some point in the discussion, Ms. Coleman would ask a student, usually the one who proposed the method, to state it verbally while she copied the words on the board. In this discussion, that's what happened next.

Formulating Chang's Method in Words

Ms. Coleman called on Harold who said that he agreed with what had just been said by Chang and Ataia. When Ms. Coleman asked him to say what he agreed with, Harold began, "That if you have 10 times any number, you just . . . " but then didn't know how to continue. Ms. Coleman turned to Chang who said that Harold had not said what he was trying to say. Chang then dictated while Ms. Coleman wrote his method on the board. With some negotiation between the two of them to arrive at the appropriate words and mutually agreed-upon meanings, Chang and his teacher finally came up with the following statement which Ms. Coleman wrote on board:

> If you multiply a number by 10, you just add a 0 the number you multiplied 10 by.

When Ms. Coleman asked if this was "clear to everybody," some students called out, "NOOO!" while others responded, "YESSS!"

Proving that Chang's Method Works

Ms. Coleman suggested that they "go on and see if it makes sense just a little bit further" and "see if this rule is going to apply." Ms. Coleman then asked Calvin what she would get if she multiplied 10 times 5. Calvin replied, "50." Following up with this problem and others to see if Chang's method applied, Ms. Coleman and the students continued:

> Ms. Coleman: If you multiply a number, does it give us that number, just with a 0?
>
> Calvin (with other students): Yeah.
>
> K: OK, 'cause you multiplied 10 x 5, it gave us that number plus a 0, it gave us 50? . . . What's 10 x 0 then? . . . Ataia?
>
> Ataia: 10 (several students gasped loudly), *0*!
>
> K: OK, You're saying 10 x 0 is 0? (She then wrote it on the board)
>
> Ataia: Yeah.

At that point in the discussion, Ms. Coleman asked if there was any disagreement with what Ataia had said. Chang volunteered that he wanted to change his method to add a sentence saying that his method would work for all numbers "except zero and negative numbers." Ataia and Melissa then agreed with Chang, but Arnie asserted that Chang's method would work for zero, because "00 is the same as 0." Ms. Coleman rejected this idea, indicating that it was an unnecessary complication, and discussion continued for more than a half hour on issues related to how, whether, and when "Chang's method" worked. Near the end of the class period, Ms. Coleman told the students to take out some scratch paper and

> write down 10 times some numbers and find out if there's a case where this doesn't work. I'd like to come around and see what you're doing. And, if I ask you, please let me know if you think this works in all cases, in fact, with the numbers that you tried.

Then Ms. Coleman circulated, looking at the students' work and making comments. Finally, she told them to stop and she asked, "If you tried numbers smaller than 200, did this method work for all those numbers? The students chorused, "Yes!" in unison. Ms. Coleman concluded, "So can we all agree that if you multiply a number times 10, it's going to give us that number, but just add a 0?" The students chorused, "Yes!" again.

Further Development of "Chang's Method"

When students came in from lunch to start mathematics the next day, they found several number problems on the board; they knew they were to begin working on these problems immediately. Ms. Coleman often began mathematics classes this way; she called the problems a "sponge" following an idea that she had gotten a few years before in Madeline Hunter-type workshop. She felt having students work on such problems during transition periods helped to "soak up any learning time that might be wasted." The problems this day were as follows:

$$\begin{array}{ccccc} 42 & 35 & 16 & 123 & 271 \\ \underline{x10} & \underline{x10} & \underline{x10} & \underline{x10} & \underline{x10} \end{array}$$

Next to the problems were the directions: "Please copy in your notebooks and be ready to explain your strategies." Ms. Coleman had created the day's sponge problems to build on the previous day's discussions about multiplying numbers by 10, and to allow students to use Chang's method as well as other strategies that they might come up with for solving these kinds of problems. Ms. Coleman had written Chang's method on a large yellow sheet of construction paper and posted it on the front board for all to see.

This day's problems were representative of the kind posed by Ms. Coleman throughout the multiplication and division unit. No contextualized problems or "word" problems were ever posed by Ms. Coleman or introduced by students; only number (computation) problems such as these were discussed. Typically, Ms. Coleman began the whole-class discussion by asking a student to explain how he or she had solved a sponge problem or to demonstrate why he or she felt a certain answer was correct. Then the teacher asked other students whether they agreed or disagreed with this solution and explanation and why. Often a student who disagreed would propose an alternative solution or idea, and the class would move to consider these new ideas.

After students had worked on these problems individually at their desks for a few minutes, Ms. Coleman handed out a Mad Minutes sheet to each student. Class usually began or ended with these short timed quizzes on "basic number facts." Mad Minutes quizzes were corrected by the students themselves immediately, and then handed in to Ms. Coleman publicly, in the order of "number right," beginning with the students who had gotten the most facts correct. The remainder of the class was then spent in whole-class discussion of one or more of the sponge problems, of other problems that came up in relation to the sponge problems, or of a method that had been proposed.

Patterns of Participation

As on the days that followed, students did much of the talking, but Ms. Coleman posed the problems, moderated the discussion, decided whom to call upon, and often used her position to steer the conversation away from what she saw as difficult areas and toward what she saw as more useful strategies and ideas. Although students talked a lot, the pattern of discourse was still mostly teacher-student-teacher-student. There was very little student-student discourse; we noted only three short examples in the entire 20 days of the unit. The final few classes in the unit deviated slightly from the typical discourse pattern. As Ms. Coleman became more concerned with "wrapping things up," and making sure that students ended up with some useful ways to do multiplication and division problems, she talked more and specifically told students which methods she believed would be most useful to them.

When we watched the videotapes and systematically analyzed students' participation in classroom discourse, we found that most students participated by talking at least once during most of the 18 days of class discussion. (Two days of the unit were devoted to informal individual evaluation.) The range of days with any recorded participation varied slightly among students, ranging from 14 to 18. However, when we looked at the quality of student's participation, we found a much greater variation: the least participating student made a substantive contribution to discussion on only 2 of 18 days, while the most participating student made substantive contributions on 12 days. On the average, students made a substantive contribution to discussion on 6 days in the unit. Making a substantive contribution included, but was not limited to, proposing any new mathematical idea or method during discussion, calling attention to necessary refinements or exceptions to a method under discussion, or demonstrating an example or a solution on the board. Agreeing or disagreeing with a student's explanation or restating a classmate's statement were considered as "participating" but not as "making a substantive contribution."

Understanding What Was Happening from the Teacher's Perspective

From Ms. Coleman's point of view, these six weeks of mathematics teaching in April and May represented the culmination of a year in which she had been experimenting with new ways of teaching mathematics that were aimed at teaching mathematics for understanding. But for Keisha Coleman, this was only another major stage in a life long task she had set herself as a learner--to improve her knowledge, skills and practices in ways that would help her be a better teacher.

When Ms. Coleman began teaching at this school 16 years ago, she had just completed a degree in elementary education from the nearby state university, and the school district was using an individualized mathematics program that was all the rage at the time. Ms. Coleman recalled feeling like she was not "really teaching" as she had been educated to do. She disliked the program because she felt that "everybody was just everywhere in the book," so all her instruction "was just hit or miss." A turning point for Ms. Coleman came when the district adopted the Comprehensive School Mathematics Program (CSMP) (Mid-continent Educational Research Laboratory, 1985), an innovative, but tightly organized and scripted, mathematics program that emphasizes mathematical concepts and problem-solving (for a fuller description of this program, see Remillard, 1990). Ms. Coleman suddenly felt like she was really teaching because of the questions that she was "constantly asking children, trying to get them to rethink or to think about their responses, rather than just giving an answer." She also liked the information in the teacher's guide and all the "workshops you could go to." She felt that teaching CSMP was the beginning of a great change in her attitude toward mathematics and mathematics teaching: "Before I had always told [my fellow teacher, Deborah Ball], 'Don't bring anything math-like my way because I'm not good at it.'" In developing her use of CSMP, Ms. Coleman became known as an excellent teacher of elementary mathematics in her district.

Still, Ms. Coleman wanted more for her students than CSMP seemed able to provide; one of her professional goals for the year was to learn more about and improve her mathematics practice. During the year previous to this study, Ms. Coleman's school was officially identified as a "professional development school" in association with the College of Education at the nearby state university, with one of its goals "to develop and put in place new forms of teaching for genuine conceptual understanding in core subject areas" through facilitating preservice and inservice teacher learning. Her school had had close ties with the university since the early 1980's; two university faculty members who are also experienced elementary school teachers, Magdalene Lampert and Deborah Ball, also taught math there. Ms. Coleman felt that over the last decade the staff at the school had been developing a real professional culture aimed at improving their own learning as well as that of their students. Keisha Coleman agreed to participate in this research study in part because she hoped it would give her an opportunity to reflect on and change her own mathematics teaching, although the direction she wanted to move was not clear until she spent a week in November of the project year observing Deborah Ball's teaching of mathematics in the third-grade classroom next door.

Keisha's Coleman's Understandings of Deborah Ball's Teaching

Keisha Coleman was struck by several aspects of what she saw happening in Deborah Ball's classroom.[5] She was particularly interested in the way Deborah's students could spend a full math period discussing only a few problems: offering conjectures, arguments, demonstrations, justifications, proofs, solutions and criticisms of each other's mathematical thinking, while "Never once did the kids turn to Debbie and say, 'What's the answer? Why aren't you telling us?'" She was also impressed by Deborah's deep knowledge of her students' mathematical understandings and beliefs, as reflected in the detailed narratives Deborah had written on each of her students' report cards. During interviews in November and January, Ms. Coleman talked about her decision to incorporate several ideas that she had gotten from watching Deborah Ball, including:

---

[5] For an expanded discussion of Ms. Coleman's impressions of Ball's teaching and the immediate influences on Coleman's practice, see Peterson (1992).

- Asking students to "prove" their answers, to explain or justify or demonstrate why they think they are correct both during class discussions and on homework
- Encouraging students to use manipulatives both when working on problems and when trying to explain their thinking to their peers
- Writing down students' "methods" for solving problems and posting them above the front blackboard, for future reference and also future reevaluation by their fellow students
- Having the children write their mathematics work in a notebook in ink, so that they could not erase traces of their thinking
- Planning for future lessons using students' ideas that have surfaced in current discussion or homework, rather than the CSMP textbook script.

Ms. Coleman's Understanding of Teaching Mathematics for Understanding

Ms. Coleman had several reasons for trying out these ideas in her mathematics teaching. First, she wanted to know more about what each of her students actually understood in mathematics, "to know how they're thinking inside." She hoped both the math notebooks and the classroom discourse would help her access her students' thinking, because, "All too often, when you have taught a lesson, and you may give the children, you know, practice sheets or something like that, . . . you never know if they have [really] grasped on to what it is you've taught." She wanted her students to listen carefully to each other's thinking, partly because she came to believe that children learned mathematics better when they heard it from one of their classmates rather than from the teacher: "They hang on to what their classmates say a lot more than any information that I could give them." She wanted her students not to be "afraid of math . . . including the girls," to be willing to offer a conjecture, suggest a method, or even challenge their own previous thinking publicly, instead of thinking, "'What does she want? Well, I'm not going to raise my hand because I don't really know . . . .'" She wanted them to realize that "you can use different ways of solving problems. There's no one correct way to do it. . . . To say, 'Yes, this is one way you can do it , but there are other ways, too.'" Finally, she wanted them

> to be able to go and search for their solutions, or go in search for answers, and not to just take whatever's being said verbatim . . . [to] think creatively or come up with methods for solving problems . . . [to] really understand . . . why this is happening and what is causing this.

Keisha Coleman's ideas of what she would like her practice to be like were similar in many ways to some reformers' ideas of "teaching mathematics for understanding." This is not surprising because Ms. Coleman's ideas were based in part on observations of Deborah Ball, one of the leading researchers and practitioners in this field who was also a co-author of the NCTM Professional Teaching Standards (1991).[6] Ms. Coleman wanted to have students talk much more, although it was unclear whether they would be talking with each other or only in response to her questions. She wanted to cease being the "final authority" in class, to help students learn to make their own competent judgments about the truth of mathematical answers, instead of always asking her for the "right answer." To aid in this process, she wanted students to use manipulatives and to advance arguments about how they could "prove" their answers. She believed in the importance of understanding her students' thinking. She wanted students to develop their own methods for solving problems, and she would post these above the board, not as necessarily "correct," but as ideas to be taken seriously. She wanted the class agenda to follow, in large part, from students' thoughts and questions, rather than from a preset curriculum. Finally, she wanted her students to become confident, intentional, self-directed learners.

---

[6] For a description of Deborah Ball's teaching from her point of view, see Ball (1993).

Ms. Coleman embraced these new ideas and tried them out in her mathematics teaching while continuing to hold views of mathematics and mathematical knowledge consistent with her past mathematics teaching. Although Ms. Coleman wanted the students to begin constructing their own mathematical knowledge, she viewed this knowledge as consisting mainly of "methods," procedures for doing problems. She understood mathematical knowledge as "information" that children would learn by listening, even if to their peers, and then needed to "hang on to," rather than as something she expected students to transform and integrate into what they already knew. She assumed that if students learned mathematics with understanding rather than rotely, then they would be able to apply this knowledge outside of school, but in her interviews with us, Ms. Coleman did not describe how this would happen.

Understanding What One's Doing; Doing What One's Understanding

Some researchers would see it as problematic that Ms. Coleman redesigned her mathematics practice without explicitly reconsidering or challenging her underlying assumptions and beliefs about mathematical knowledge and about the learning of mathematics (see, Brown & Borko, 1992; Kennedy & Barnes, 1993). However, other researchers suggest that successful reforms can occur when teachers have opportunities to do exactly what Ms. Coleman did--try out new practices and then develop conceptual underpinnings to go along with these new practices (Fullan, 1985). Understanding what one is doing or doing what one is understanding might be seen as contrasting perspectives, or they might be seen as reciprocal--two sides of the same coin. Exploring these issues became central to our cases analyses not only of Ms. Coleman as the teacher, but also of each of the five students as learners in Ms. Coleman's mathematics classroom. We explored not only what each learner did during mathematics class, but also how learners understood what happened in mathematics class and the mathematics that they learned during those six weeks.

Learners' Year-End Understanding of Mathematics and of Being Good at Math

By year's end, Ms. Coleman's students had learned a lot about division and multiplication. Eleven of 15 students were able to verbally define multiplication, mostly using the Euclidean model of repeated addition. Twelve students gave an acceptable definition of division, with 9 students giving a measurement-type definition of division using some variation of "seeing how many times one number can fit into another." One child saw division as partitive, estimating the number in each group and "counting up" the number of times indicated by the divisor to see if she reached the dividend. Two other children defined division as a multiplication problem with a missing factor. Most of Ms. Coleman's students seemed to understand the reciprocal relationship between multiplication and division, often citing the related multiplication fact in solving or proving a division problem or using multiplication-type strategies to solve division problems. Children confused the two operations only rarely--12 times during all the problems solved during 15 interviews, and only 5 times did they confuse multiplication and division operations without correcting themselves. Harold, Calvin, Selvaranee, Ataia, and Aurora each confused multiplication and division once without revising.

As part of the June interviews, we asked students to do twelve computational problems: five in multiplication, ranging in difficulty up to 4 x 12 and 8 x 100; and seven in division, ranging up to $11\overline{)132}$ and $4 \div 8$. The average score on these twelve problems was 8 correct, with a range from 3 to 12. We posed the same two word problems as on the initial interview, and this time 12 out of 14 students answered the simpler problem correctly (one student was inadvertently not given this problem), while 9 of the 15 were able to figure out the more difficult problem.

Flexible Strategy Use

Most significantly, in solving all these problems the students demonstrated flexible use and understanding of a wide variety of solution strategies. Learners used a total of 15 different strategies (descriptions of these strategies are presented in Appendix C). The average number of strategies tried per student was 7.1 and the average number used successfully was 5.9. Many children also demonstrated their ability to use more than one strategy on a problem. All in all, Ms. Coleman's students demonstrated an effectiveness and flexibility of strategy use unusual for third graders.

Interestingly, the traditional multiplication and division algorithms were not particularly helpful to these students; only four students used the multiplication algorithm successfully on any of the four multidigit multiplication problems, and only three students used the long division algorithm successfully on one or more of the multidigit division problems. By comparison, eight students used other strategies successfully on these division problems. Strategy use was closely related to achievement, with the number of correct solutions correlating at .77 with the number of strategies attempted and at .89 with the number of strategies used successfully. Looking at the scatterplots of these relationships, it was clear that even the most successful students did not tend to narrow themselves down to just a few strategies, but rather relied on a range of strategies, choosing the one(s) they felt were appropriate to each problem.

As one might guess from the flexible strategy use that students demonstrated, most of Ms. Coleman's students had developed an active approach to solving mathematics problems. The average student attempted between 10 and 11 of the 12 computational problems we posed, even though some problems were well beyond a third-grade level of difficulty. Many students persisted, made repeated attempts, and tried several different strategies even on problems that they found difficult.

Valuing Understanding and Changed Beliefs About Understanding

Although most students (11 of 15) still mentioned speed and accuracy as major criteria for "being good" at math, 8 students also valued more understanding-based and discourse-related abilities--criteria that they had not mentioned at the beginning of the year. For example, Arnie named Yan as good at math because "He answers a lot and he gives good explanations." Calvin named a friend of his outside the class, because "he helps me on the hard problems." Josephia thought that four of her classmates were particularly good because they gave her "new ideas to do math." Some students, at least, had developed new understandings *about mathematics* as well as new understandings *of mathematics* through participating in Ms. Coleman's redesigned class. Harold, whose story forms our first case study, is one student who seems to have done so.

Harold's Story: Learning About and Through Discourse

Learning to Talk and Talking to Learn

During the multiplication and division unit, Harold stood out as a student who demonstrated remarkable facility and flexibility in the ways he participated in the mathematical discourse. Harold participated regularly in discussion, talking on 17 of the 18 discussion days in the unit, and making substantive contributions on 7 days. He was most notable for his willingness to revise his answers, and to pursue a concept until he understood it. For example, during the second week of April, Harold at first proposed that 417 ÷ 10 = 41, and several students agreed with him. Subsequently, Yan suggested that the answer was actually 41.7, "because it has extra," and Harold revised his answer to agree with Yan's. Harold stayed with his revision even when Ms. Coleman expressed some surprise that he would revise, "after so many people have agreed with you." Again, the next week, Harold started to disagree with a method on the board, but then stopped in midsentence and said, "Can I revise?" Ms. Coleman said, "Sure," and Harold went on to agree with the method as stated originally. In reply to a question from Ms. Coleman four days later about any patterns that students saw in their homework problems, Harold volunteered that he had gotten "a lot wrong, because . . . I messed up and used it [Marta's mother's method] when I wasn't dividing by 10." He had checked back in his notebook and now realized the method only applied to dividing by 10. In an interview following class that day, Ms. Coleman said she "thought what Harold said today was really very good" because he was able to see his error and was willing to speak so openly about it.

We were able to watch as Harold gradually gained understanding of multiplication and division. He was not a student who started out knowing how to do multiplication and division already. As his teacher noted in early May, Harold often didn't do "too well" on practice sheets or homework, but "the more discussion, the more discussing of ideas you do, the more he catches on." For example, in mid-April, the reciprocal relationship between multiplication and division was a focus of the discussion. Aurora had proposed that you could solve 40 ÷ 5 by "doing it backwards, like □ x 5 = 40." Harold raised his hand, saying "I don't understand," and Ms. Coleman reiterated an explanation Yan had given. Later that day, Ms. Coleman asked Harold how he would check 125 ÷ 10 = 12, and he said he would multiply 12 x 10, but was not able to carry this thinking through to an answer. During the next week, Ms. Coleman proposed that Aurora's method, Chang's idea that "you can check division by multiplication," and Calvin's assertion that "division is like times," were all related, but Harold disagreed. Two days later, in an in-class interview with Ms. Coleman about some homework problems, Harold said he was sure that 400 ÷ 8 = 40 was correct, because he could "check it." He then wrote "40 x 8 = 400" on his paper, but when challenged, could not explain how he got that answer, just that it "should be."

By the first week in May, Harold showed a better understanding of these relationships. He suggested that the class could check 4 x 8 = 32 by doing 32 ÷ 8. The next day the discussion returned to this idea, and Aurora said it was like her idea, giving the example that 8 ÷ 4 = 2 could be done "backwards" by saying 2 x 4 = 8. Harold again objected to the "backwards" language, but was able to be more specific this time, saying that 8 ÷ 4 = 2 "backwards" would be 2 ÷ 4 = 8. Ms. Coleman suggested the class needed to define what they meant by "backwards," and urged them to be more specific about "what to do." Finally, Harold proposed this revision: "You can multiply the quotient and the divisor, and it will equal the dividend." Ms. Coleman accepted this, and labeled the amended method as "Aurora/Harold's method," suggesting that the kids copy it down in their notebooks. In an interview after class that day, Ms. Coleman said that Harold's idea was "really pertinent . . . I got really excited." The following day, Harold quickly completed an end-of-class assignment to "prove" 38 ÷ 2 = 19, copying the problem and then writing "19 x 2 = 38" and "19 + 19 = 38" in his math notebook.

Harold at Year's End

In the June interview, Harold showed a good understanding of the reciprocity of multiplication and division. He decided that 60 ÷ 10 = 6, "because when you check your multiplication problems, you use division" (This was in reference to the previous problem, which was 6 x 10, and which Harold had answered correctly as 60.) He also offered as 60 x 10 = 600 as a proof of his solution to 600 ÷ 10. He had developed a number of other strategies as well for solving multiplication and division problems, using seven different strategies on the nine problems he solved successfully during the interview. For four of the problems he demonstrated two different ways to get the answer, and on one problem (600 ÷ 10), he volunteered three applicable strategies. His strategies were not primarily algorithmic, ranging from direct representation using "sticks" (tally marks) and base-10 blocks to "Chang's method," which stated that you can multiply any number by 10, just by adding a 0. Harold's answer of 13 to 132 ÷ 11, although it was incorrect, reflected a growing sense of the size of numbers and some facility at estimating. He explained, "I knew that 10, 10 x 10 is a hundred, so maybe 11 times was 120 or something, and 11 another time would be 132." His learning was recognized by at least one of his classmates, who named Harold along with Chang, Yan, and Ali as the best in the class, "because they give me new ideas."

Harold had also changed many of his ideas about math. He said that this year math was his favorite subject, and the part he liked best was "discussing stuff with the class," even though explaining was "hard sometimes." If someone disagreed with him in class, "If they say why they disagree, I might think that they are right, and revise the thing I did," which he said is "OK" to do. Harold no longer looked solely to the teacher for answers and instruction in mathematics, perhaps even carrying this shift to extremes when he told Ms. Coleman on the final day of the unit, "I don't think that you taught us about [multiplication and division]. I think that the other people in the class did, because you weren't the person that came up with the methods and stuff to do it."

Harold's facility with mathematical discourse and ideas came through perhaps most clearly during the small-group problem-solving interview we conducted after school was out. Harold solved all three word problems easily, working closely with one of the other group members, Ali, to figure out the wording. During the early part of the interaction, Harold repeatedly tried to draw out the other, nonparticipating student, asking him "Calvin, do you think it's A and B?" and later, "What do you think it is?" When Calvin expressed a different opinion, Harold asked, "Why do you think that?" and really worked to follow his reasoning as Calvin fumbled for an explanation. When Calvin offered to change his answer, "because you want me to," Harold indicated that he did not want Calvin necessarily to change his answer, "We're just asking you why you think it's [that]," and later told him, "You shouldn't just keep changing [your answer] because someone disagrees; you should stick to what *you* think."

Understanding Harold

Harold learned a lot in Ms. Coleman's math class this year. Not only did he learn a lot of different ways to do math problems, he also learned a lot about how to talk and think about mathematics, seeming to have achieved many of the goals that Ms. Coleman had in mind when she adopted her "new" style of "teaching for understanding" in math. Harold demonstrated a disposition to listen carefully to and learn from his peers; he explained his own ideas clearly, and was willing to risk saying "i don't understand" and to learn publicly from his own mistakes. He was obviously comfortable using manipulatives, as well as a variety of other means, and could use them effectively to solve problems. Above all, he seems to have confidence in his abilities, and to be active, curious, and self-directed in learning and doing mathematics.

Harold is a case of a student who developed substantially in mathematical understanding over the course of the mathematics unit and whose understanding of the purposes of the classroom discourse and the mathematical tasks seemed to by in accordance with his teacher's. By contrast, Calvin is a case of a student whose understanding of both the process and the subject matter in mathematics class often seemed to be substantially different from that of his teacher and classmates.

Calvin--In a Different Place

During the end-of-the-year interview, Calvin gave a description of himself that captured well his behavior during mathematics class: "I am usually out in 'dreamyland.'" When Calvin was called on during mathematics class, he frequently did not know what problem the class was discussing, much less which solutions had been proposed and how they had been justified or critiqued thus far. Calvin's demeanor when answering often seemed to indicate a feeling of being elsewhere, or at least wanting to be elsewhere--he usually had his head lowered, his eyes on the floor, and, often after a long pause, he would mumble a brief or disjointed answer. A typical exchange occurred the second week in April when the class was discussing the meaning of 417 ÷ 10. Ali had just proposed that it means "you have to find out how many 10s are in 417." Ms. Coleman then asked Calvin what he thought about what Ali had just said. Looking down at his desk, Calvin replied, "I agree with him." Ms. Coleman queried, "Why do you agree?" There was a long pause before Calvin finally said that he "just agreed with him." His teacher persisted and asked Calvin what Ali had said that Calvin agreed with. Again, there was a long pause before Calvin finally mumbled something. Finally, Ms. Coleman asked Calvin if he knew what problem they had been discussing. Calvin looked up at the board and after another long pause said, "305 divided by 50?" At that point, the teacher called on Josephia, who stated correctly that the problem they were working on was "417 divided by 10."

In her interviews with us, Ms. Coleman often expressed concerns about Calvin, saying "Sometimes, I really feel like I'm losing him," although "he's really a lot brighter than you see." On another occasion she volunteered that she thought "a lot of times that things are just going by [him]." She had noticed one day, for example, that "Calvin could do the calculation (a subtraction problem) properly, but he couldn't tell you what he did." On another day, Ms. Coleman said, "I just don't know what I'm going to do with him; he's out to lunch!" Ms. Coleman and Calvin seemed to agree that Calvin's mind was often simply somewhere other than on the mathematics and the mathematical discourse in class. One way that Ms. Coleman saw the problem was as a lack of attention. Yet when she concentrated on getting Calvin to pay attention, her efforts met with only modest success.

The Problem of Calvin: Take 1--Engaging Attention

Ms. Coleman frequently called on Calvin to respond during class discussion even when he was not volunteering. As a result, Calvin actually spoke during class discussion on 16 out of 18 days. Calling on students who did not have their hands raised was one strategy Ms. Coleman used frequently to try to hold students' attention during whole class discussions. She often reminded the students that "I'm going to be calling on you during math time, so you really need to listen." On April 11, Ms. Coleman explicitly explained this teaching strategy to the class, saying:

> You must know that I call on you when I don't think you are paying attention a lot of times. But that's not the only reason why I call on you, but sometimes I do. So you need to make sure that you are fully aware of what people are saying, so you can repeat it. If you repeat it, then, that at least let's me know that you are listening to what your classmates are saying, OK?"

The students were well aware of their teacher's technique for ensuring their attention. During a class discussion about the reasons students should listen to each other, four students volunteered that students should listen because they might get called on and not know what had been said. Interestingly, during this discussion, Ms. Coleman did not seem satisfied with this response, repeatedly pushing students to come up with more "positive" reasons.

With Calvin, and sometimes with other students who were having trouble, Ms. Coleman would take this strategy one step further, often reminding the student to pay attention before an explanation was given, but this strategy was not usually any more successful in gaining the student's involvement. In addition to being reminded to pay attention and being called on frequently, Calvin was by far the most often disciplined student. For example, on one day in early April, Calvin had to erase his name from the board (a consequence in the Assertive Discipline program Ms. Coleman was using) for forgetting his math notebook. The next day he waved at the camera, and as a result, Calvin lost his star (one step worse than having to erase one's name in Assertive Discipline). Later that day he was asked if he would like to spend the rest of math time in the office. Calvin frequently forgot or misplaced things--homework, signed papers, his math notebook. In fact we lack complete data on Calvin because he lost his math notebook (again!) at the end of year before we had a chance to photocopy it.

The Problem of Calvin: Take 2--Understanding and Valuing Calvin's Ideas

Late in April, Ms. Coleman seemed to take a different tack with Calvin. The students were discussing Marta's Mother's method, which they had constructed as "You can divide any number by 10, just add an R for remainder, like ☐ R ☐." Ms. Coleman again called on Calvin, who was not participating.

Ms. C: Calvin, do you have an opinion?

Calvin: No.

Ms. C: No opinion? Everybody has an opinion. What do you think about what it says right there? I have a feeling that we are still not focused in, some of us, and we need to be. I'm sorry to say it again.

Calvin: Umm . . . I quite agree.

Ms. C: You quite agree with what? (Calvin said something inaudible; his head was down toward his desk.) You think what?

Calvin: That you can divide any number by 10 . . . (Calvin's voice faded out into silence.)

Ms. C: (quietly) That you can divide any number by 10, just add an R, like, for remainder?

Calvin: (looked up, flashed a smile) Yeah!

This dialogue began the same way as had many others between Ms. Coleman and Calvin. Once again, Calvin was called on when he had not volunteered to talk, and once again, he was unclear about just what had been going on. However, this time, instead of letting him flounder and mumble until his confusion became painfully apparent, Ms. Coleman completed Calvin's statement for him, allowing him to agree with it. Calvin's relief was evident, and he responded with a big smile. More importantly, only 10 minutes later, Calvin volunteered for the first time during the entire unit, advancing the idea that "division is just like times." Ms. Coleman encouraged him, saying, "I like that idea, can you give me a little bit more, or an example." Calvin got bogged down trying to explain what he meant, and Ms. Coleman drew in other children to fill out the idea. The resulting co-construction was labeled "Calvin's method" and included the example that "40 ÷ 8 = 5 and 8 x 5 = 40." As she wrote this on the board, Ms. Coleman said, "I like that; thank you, Calvin."

During the next day's class, Calvin again volunteered, this time to discuss Chang's solution of 45 to the problem 450 ÷ 10. Calvin maintained that the answer should be "bigger" than 450, and Ms. Coleman spent nearly 10 minutes in class trying to untangle why he thought this. At the conclusion, she asked Calvin to try the problem 450 x 10 on scratch paper, and then said to the rest of the class, "I know we took a little time with Calvin, but that's OK because he needed some extra time with that."

Calvin volunteered twice in class on the following day, and in an interview following class on that day, Ms. Coleman talked enthusiastically about Calvin's increased participation:

> I was really impressed with Calvin, because I just asked the question, "Do you see any patterns or relationships?" and he was the first one with his hand up! "Well, division is just like times." It was like, Wow! . . . I wanted to jump up and down.

Yet Ms. Coleman was still quite concerned about him, mentioning that his mother had called the previous night because "she could see that he was struggling."

Retake 1--Lack of Attention

Calvin's new pattern of involvement did not seem to last. Three days later, when Ms. Coleman called on Calvin again when he had not volunteered, he had trouble restating a main idea that was actually written up on the board in front of him. The next day Calvin reported losing his math notebook, and, when called on, was twice unable to comment on or even restate what had just been said in class. In an interview that week, Ms. Coleman described Calvin as "in and out, in and out." Regarding a statement he had made in class that $12 \div 2 = 24$, Ms. Coleman said

> I just thought he wasn't tuned in. He just kind of heard a piece of it and was repeating that piece. Sometimes what I do is I go back and I will ask him what was my question . . . and sometimes you can see in his face, when he's not quite sure . . . so I know that he knows that he hasn't given the response he knows he needs to give. But because he wasn't really that tuned in, he doesn't quite have it.

Again, Ms. Coleman mentioned that Calvin's homework was usually quite good,

> I don't know if someone is sitting down at home with him or what. . . . Calvin sometimes gets [the answer], but isn't able to say to you, "Well, this is how I got it." Or he'll just say . . ."Because my mother said so," you know, that's his famous line.

But when Ms. Coleman called on Calvin, to draw him "in," it only seemed to accentuate how much Calvin was "out." On May 4, Calvin could not restate something Harold had said, even after the statement was repeated twice. Later that day, Ms. Coleman tried to focus him on a reply from another student:

> I want you to listen, Calvin. You know I can always tell when you're, when you're daydreaming or your mind is someplace else; you get this look on your face. All right? Now, he's talking about something that *you* said, so you need to be focused in so you can say, "Yes, I did say that" or "No, I didn't." Calvin, you always gotta be with us, all right?

At the end of class later that week, Calvin worked on a short assignment for some time before he discovered that he was working on the wrong problem. Ms. Coleman told him, "A perfect example of your not paying attention, Calvin. You don't even know what to write down. You miss a lot of what goes on in the classroom, Calvin, just like that."

Calvin at Year's End

The end-of-the-year interview revealed that Calvin had, indeed, missed a lot. Of the multiplication and division number problems we posed on the June interview, Calvin was able to solve correctly only 5 out of 12 problems, and he used only four different strategies successfully. He was able to do 4 of the 5 multiplication problems, primarily using the traditional algorithm, but he refused to try most of the division problems, especially those with large numbers. Calvin explained that division was the hardest thing for him because he kept "thinking division is just like times, but it is, and each time I keep timesing, and putting the wrong answer." He did seem to confuse division with multiplication sometimes, as when he maintained that 12 ÷ 2 would be the same as 12 x 2. Interestingly, this "confusion" may also be viewed as a first stage in Calvin and his classmates' developing understanding of the reciprocity of multiplication and division, as it was Calvin who had proposed the idea that "division is like times."

When Calvin relied on his own ability to make sense of a problem rather than on rote memory of an algorithm, he did better. For example, Calvin solved the "marker" problem quite readily, counting out six groups of four on base-10 blocks, although he had been unable to solve this word problem at the beginning of the year. He also used base-10 blocks to solve two other problems (326 - 99 and 46 + 28) which had stumped him in October.

During the June interview, Calvin indicated that math was no longer his best subject, saying that he liked math "just a little" and that he particularly disliked Mad Minutes because, "I usually never get an A+; one time I did, but that was my only time," although "lots of kids" got A+s. Calvin seemed to have a fairly accurate picture of his own achievement; his average on Mad Minutes for the year was only a D+, well below the class average of B-.

When asked if math was hard, Calvin said, "No, when you concentrate, it isn't hard, but if you don't concentrate, it is hard," and went on to describe his difficulties:

> I'm usually out in dreamland. . . . it's just that I kinda think that when we get off in that discussion, I say, "Gee, this is boring . . . why can't we get onto the other stuff instead of the, um, this boring discussion." [I mean] stop the discussion and go on to the next problem or go on to the next stuff we have to do, like science or stuff like that.

Calvin perceived some differences in classroom discourse this year, saying, "If someone wants to disagree with the other person, they would show why and prove how," while last year "they would have a vote to see which one is right," but he said he preferred the voting.

Calvin's impatience with discussion also showed during the small group problem-solving interview in June. He did not participate at all in the first eleven minutes of the discussion between Harold and Ali, except to respond to two direct questions by saying, "I agree" and "OK." The other two students, Harold and Ali, came to an agreement, which they believed Calvin shared, but while they were reporting their conclusions to the interviewer, Calvin spoke up and said, "Not me!" The others asked him to explain what he thought, and Calvin talked about the context of the problems which were as follows:

> A. Marla has a job after school. Last week she worked two hours and earned $10.50. How much did she earn per hour?
>
> B. This week Marla worked two hours and earned $10.50 per hour. How much did she earn this week?

C. Marla worked two jobs. She earned $5.25 on the first job and $10.50 on the second job. How much did she earn at both jobs together?[7]

Since Marla was in all the problems, Calvin assumed that these were all true statements about Marla and her jobs, and he tried to make sense of these as real-world problems. For example, treating these as authentic problems, Calvin voiced concerns about whether the woman was going to school and whether she could work all three jobs without a break. When the other students continued to push for his reasoning, Calvin began to answer in a seemingly random fashion, changing his answer at least six times in as many minutes. When they asked him why he was changing his answer, he said, "I kept changing my answer 'cause you wanted me to. . . . I gave you the answer, and you kept asking *why*!" Ali explained, "We're not telling you to change you answer," and Harold added, "We're just asking you why you think that." Calvin then said, "I still think my answer is right." Finally, after further attempts by the other students to understand his thinking, Calvin bargained, "You guys don't like the answer, so I'll change it. . . . I'll change my answer, and you guys won't ask why." He would not explain further, for the next five minutes simply saying, "I agree." Calvin seemed to think the group's task was not to explore and discuss possible similarities and differences among the problems, but rather to agree on some "right" answers as quickly as possible.

This interpretation is consistent with some of the ideas about mathematics and mathematical discourse Calvin expressed during his individual interview in June. He said that Chang and Yan were probably the best in his math class, because they always have their hands "up in the air"; and he believed they got to be good at math "by practicing." He also said that his friend Joe was good at math, because he could do "the hard stuff" on Calvin's homework. He disliked having people disagree with him in class, especially, "If the answer's right, and they don't agree, it's not fair!" When asked to define a method, Calvin said, "It's not usually right, but sometimes they use it a lot . . . it's just like a little guess." The only method Calvin could remember was his own--"division is just like times."

---

[7] This question was one of the "open-ended" items developed by the California Department of Education (1989) to be given as part of the CAP (California Assessment of Educational Progress). The item was intended to be given to high school students, but in this study we found that many of Ms. Coleman's third grade-students were able to solve the three word problems correctly and engage in interesting discussion around them.

Understanding Calvin

What happened to Calvin in this mathematics class and why? Ms. Coleman's ideas, our observations, and Calvin's own explanations all seem to agree that, during much of the class discussion, Calvin was simply in another world of his own. Calvin gradually fell behind because mathematics learning in this classroom depended on participation in the world of the classroom, including listening to and understanding the discourse of Ms. Coleman and his fellow students. Ms. Coleman characterized Calvin this way:

> He's a very, very bright young man, very. . . . I'm hoping that, you know, he'll be able to catch up one of these days, but he'll miss a lot because he can't really deal with the routine of things, you know, the everyday, day-to-day kinds of things. I think he'll miss out. . . . When we were working with division, that was something that was very new, and he wasn't paying attention and wasn't listening and, you know, was off doing his own little thing, and when he got finally got the practice work, he was just lost. He was just absolutely lost.

Perhaps Calvin did not attend to discussions partly because he found them boring, but he may also have found them painful. His inattention, followed by Ms. Coleman's calling on him to compel his attention, may have set up a vicious circle in which, each time he was embarrassed by not knowing how to reply, he withdrew further from the discussion. When Calvin's mathematical ideas were explored and supported in several later class sessions, he responded happily and showed a greater willingness to volunteer and participate than he had previously.

Calvin may also have tuned out during class discussions partly because he had a different understanding from that of Harold and Ms. Coleman about the point of these discussions. For Calvin, mathematics continued to be a matter of getting the right answers. If the teacher no longer told "the answer," he thought it was best to vote or even just "agree" quickly during discussion; whatever it took to get "the answer" and "go on to the next problem." Calvin did not seem to value knowing multiple ways to solve problems; he seemed to prefer using the traditional algorithm whenever he knew it, and to him all the "methods" talked about in class were just "little guesses." He frequently used information from his mother or a friend to complete his homework, but then he was unable to explain how he had done the problems. Although Ms. Coleman was trying to teach math "for understanding," Calvin had a different view from hers about what that "understanding" entailed; he certainly did not think of discussing, agreeing, disagreeing, and comparing different strategies and interpretations as ways to build the understanding he would need to "go on" in math. Perhaps it might have helped Calvin if the ideas underlying the discourse-based teaching that Ms. Coleman adopted had been more explicitly discussed in class, or perhaps not--perhaps he wouldn't have listened, figuring it was just more "boring" discussion.

The Problem of Calvin: Take 3--Entering Into Calvin's World

At the end of the year, Ms. Coleman was clearly still wrestling with what to do about Calvin. She repeated her belief that Calvin's problem was primarily lack of attention, rather than lack of ability:

> Anything that's special or different, you can always have his attention. I wish I could think of something special and different to do every day, 'cause that's the only way you get him involved, the only way. . . . [It's] not because I don't think Calvin knows. I think he doesn't pay attention half the time, but I think he has an idea of what it is he's doing.

In this quote, Ms. Coleman provided another way of thinking of Calvin--as a gifted child who lived in a world of his own construction. Ms. Coleman speculated that a way to reach Calvin might be to have him learn within the context of projects and things that would interest him in his world. Ms. Coleman elaborated on this idea in a later interview:

> I've always said to Stan (her co-teacher), "You know, that Calvin is a bright guy," and Stan said to me, "You know, Calvin is a genius." And he is, but, you know, a lot of times people who are really, really talented or

> gifted, are so unorganized they can't, you know, keep things together, and they lose things all the time. You know, Calvin has all that, because you give that guy a special project, and he is just, he is just amazing, but the every day, uh, routine kind of things, he can't handle at all. . . . [When] we did the special science lesson, and that was videotaped, Calvin was outstanding . . . anything that's different and special, he just shines.

Ms. Coleman reported in October 1990 that Calvin's mother had sent him to a local parochial school, known for its emphasis on academic "basics" and "tough" discipline. Although, during the unit we observed she herself had sometimes seemed to pursue a rather coercive strategy to gain his attention, Ms. Coleman did not now think this was a good idea for Calvin. She said,

> I know they must probably be having a heck of a time. She called me this summer to find out whether that would be a good recommendation and I didn't recommend it for him and I think she also talked to Stan and he also did not recommend it, but she sent him there anyway, . . . She thinks that if he's given more work and that if the school could be harder on the children--see that doesn't work with Calvin. Calvin is a very bright young man but he needs variety. Now I think he probably would do a lot better in my class this year, I do, because it's not, I mean we're not doing things traditionally or normally.

Calvin, in fact, did not do well at his new school either, returning the next year to Ms. Coleman's school for fifth grade.

Calvin was in some ways very different from Chang, a fellow student whom he greatly admired, in that Chang will probably do well throughout his elementary school career. Yet like Calvin, Chang developed different understandings from those of Harold, even though they had all participated in the same mathematics classroom.

Chang's Story: Methods or Understandings?

In January, when asked which students she thought understood odd and even numbers, Ms. Coleman named Chang first. In March, when asked which students were strongest in their understanding of place value, Ms. Coleman again named Chang first. The interviewer then asked, "Why do you think he understands?" and Ms. Coleman replied, "Because Chang comes up with all the conjectures and the methods . . . in fact, most of them, I think, that we've had have been his."

The Making of a Method

Chang continued to play a leading role during the multiplication and division unit we observed in April and May. He participated during all 18 of the class discussion days, and made substantive contributions on 12 days, more days than any other student. Of the nine student-originated methods posted in front of the room by the end of the unit, three were Chang's. One method suggested that you could check division by multiplication, and another involved doing division by serial subtraction. The third was the one known as "Chang's method," and was Chang's most notable contribution to the classroom knowledge base created during the unit.

Chang's method originated on the first day of the unit during the discussion described earlier in this chapter and was written down as "If you multiply a number by 10, just add a 0 to the number you multiplied 10 by, except for 0 and the negative numbers." The next day Ms. Coleman gave students a "sponge" of five problems, all with a two- or three-digit number multiplied by 10. She intended to follow up on the development of Chang's method, but of the five students who volunteered to demonstrate their solutions, only one--Chang--used "Chang's method." Chang's work in his math notebook also showed that he used his method on all five problems. As Chang demonstrated his solution to 35 x 10, he elaborated his method

> Chang: (He had written 35 x 10 vertically on the board.) I crossed off a 0 from the 10. I put it behind 35, and I moved the 1 (he indicated that the 1 from the original 10 has moved right, to the units place where the 0 used to be), and I multiplied 1 times 0 equals 0. 1 times 5 equals 5. 1 times 3 equals 3 (following the pattern of the traditional multiplication algorithm). And the answer is 350.
>
> Ms. Coleman: So why do you have to move the 1?
>
> Chang: Because if you put it over here (in the tens column), you would have to do this, you would have to go backwards . . . and you do, you go from the right to the left, you don't go from the left to the right.
>
> Ms. Coleman: I want you to explain that one more time. . . . And try to use as few words as possible. You're putting a lot of words in, and it's kind of hard to follow you. So say exactly what you mean.

> Chang: I started out, it's 35 times 10. All I did was cross out the 0 behind the 1 and put the 0 behind the 35. And then I moved the 1 to under the 0 . . .
>
> Ms. Coleman: OK. So if you move the 1, erase the 1 that's already there. Now, tell me, once again, why did you move the 1?
>
> Chang: Because in [multiplication] you're supposed to go from the right side to the left side, you're not supposed to go from the left side to the right side.

In the above example, Chang responded to his teacher's request to explain his thinking by continuing to restate his procedures. An eight-minute dialogue between Chang and his teacher ensued, during which Ms. Coleman tried to persuade Chang that he didn't need to "move the 1". Finally, Ataia said she understood what Chang was saying, and she volunteered:

> Ataia: I think I know what he's saying. He said, he put the 0 there, and he says it'd be going backwards. 'Cause I think he means you start from the ones and then you go to the tens, hundreds, thousands . . . (Ataia mimed doing an algorithmic multiplication process.)
>
> Chang: Yeaaaah. That's what I saaaaid! (He sounded exaggeratedly weary)
>
> Ms. Coleman: Yan?
>
> Yan: I disagree with Chang.
>
> Chang: Great! (sarcastically; some of the other students laughed.)
>
> Ms. Coleman: It's OK if someone disagrees with you, Chang, isn't it?
>
> Yan: (began an inaudible explanation)
>
> Chang: Yan, can I cut in on you? Ms. Coleman is asking me why I put a 1 there. I'm trying to let her know why I put the 1 there.

The dialogue between Ms. Coleman and Chang then resumed for several minutes, with Ms. Coleman becoming more insistent that Chang need not move the 1, and Chang becoming increasingly agitated as he failed to convince her of the necessity. Finally, Ms. Coleman said

> Chang, listen to me. Just answer my question. If you decide that you're going to move this number, this 0 right here . . . OK, it's, it's wrong in your mind, then, but I guess, if you're going to use a strategy like this Chang, I guess I'm just trying to think about, in terms of who you're teaching and talking to, you know, you aren't helping your classmates perhaps to maybe use a strategy because this is the one you came up with.

> OK? And I guess in terms of explaining that to them, maybe the simplest way to do that, OK? I mean, you can eliminate some steps with that.

The class ended shortly thereafter.

Chang's method was not mentioned during the following four math classes, even though the sponge exercises all involved multiplying or dividing numbers by 10. On April 17, however, Ms. Coleman asked all the students to check the solution to 417 ÷ 10 by doing 41 x 10 at their desks . After she observed several students' working, she commented, "All right; I'd forgotten about that," and she called the class to attention and asked Chang to come up and show his method again. Then she said,

> As I walked around and listened to people, that was the method that they used in order to figure out this problem. And everybody that I spoke to that used that method did an excellent job of explaining it to me.

There was some discussion about the left over 7 being called a remainder, and then Ms. Coleman asked the students to do 125 ÷ 10, 740 ÷ 10 and 741 ÷ 10 in their notebooks. Again, she walked around observing, and then called the class to attention and asked them why they weren't using Chang's method. Chang said, "You can't use that method on division."

Ms. Coleman then demonstrated how they could check their division problems using Chang's method. It seems that she believed they could use the reciprocity of multiplication and division, which they had discussed somewhat earlier in the unit, to solve these division problems, but the class did not seem to catch on, including Chang, who continued to maintain that his method only applied to multiplication. This was particularly interesting because in Chang's math notebook earlier, he had actually solved 420 ÷ 10 to get 42, and had written a note that, "I just took a 0 off 420, and there are 42 tens in 420." At the end of class, Chang showed something to Ms. Coleman, who asked him to write it down, and she would ask him about it tomorrow in class. In his notebook for that day, there was written, "If you have a number that has 0 behind it and it is being ÷ by 10, just take off the 0 from the number your(sic) ÷ 10 by."

The next day, Chang did not get to share whatever he had showed Ms. Coleman, but Arnie, when asked how to solve 40 ÷ 10, said he "timesed 10 four times and came up with 40, . . . and then I took the 0 away from 40, and looked to see if I would have the same answer." Ms. Coleman asked him whether he had tried this with any of the other problems, and when he said not, she told him "instead of crossing out the 0 in your 40, just forget about that, and tell us what you did." Right after that, Chang said he got his answer "by crossing out the 0s in 10 and 40." Ms. Coleman said, "I'm not going to expand on that right now, and ask children to comment, because I don't think they are ready yet."

Chang did not get to share his idea until four days later when, reading the statement from his earlier notebook entry and using the example 450 ÷ 10 = 45, he shared what he thought of as a new method, one for dividing by 10. The "add-a-0" method for multiplication and the "cross-off-a-0" method for division were subsequently referred to by students indiscriminately as "Chang's method." The class discussion moved quickly to focus on Calvin, who thought the answer to 450 ÷ 10 should be "bigger" than 450, but Ataia, Aurora, and Yan all had a chance to agree with Chang's new method, Ataia pointing out how it was related to his old "add-a-0" method.

Neither method was brought up in class discussion again, but during in-class interviews with Ms. Coleman, Harold, Ali, Bert, and Josephia indicated that they had used it to solve two division problems (60 ÷ 10, 600 ÷ 10) on a worksheet they had been given. However, Bert's first answers were 600 and 6000, showing that at first he was unsure whether to add or remove a zero, although he did settle on the correct answers in the end. Harold also misapplied this method to 40 ÷ 8 and 400 ÷ 8, claiming that you could just "take away a zero" from each dividend. Aurora cited Chang's method as a justification for saying that 60 x 10 = 600, which was how she solved 600 ÷ 10, by saying _ x 10 = 600. In a few students' math notebooks, there was also one set of problems dated later, in June, that seemed to involve multiplying one- and two-digit numbers by 10, 100, or 1000, but there was no indication what method(s) they used to solve these problems, and we were no longer observing daily at that time.

Understanding and Using Chang's Method at Year's End

On the final interviews, six students (Harold, Selvaranee, Aurora, Josephia, Arnie, Marta, and Yan) successfully used Chang's method to solve 6 x 10. Josephia's explanation of how she solved this problem was typical of their answers: "Chang said if you do ten times six (she wrote 10 x 6) equals 60 (she wrote = 60) because you take off the zero (crossed out 0 in 10) and add it there (wrote 0 next to 6)." Four children (Bert, Ali, Marta, and Yan) extrapolated Chang's method to solve 8 x 100, giving explanations like this one of Ali's:

> Ali: . . . it's just like 10 eight times, and that would be 80, so what I figured is, if it was 100 times 8, it would be just like 800 . . .
>
> Interviewer: Ten times 8? How is that like it?
>
> Ali: It's, it's, it's almost, um, except for, um, this (the 1) is all the way 'til the hundreds column.

Three other students (Josephia, Melissa, and Selvaranee) used what seemed to be related place value strategies on this problem. Five students (Josephia, Selvaranee, Chang, Bert, and Yan) used Chang's method to solve one or more division problems, and Ataia tried to use the method, but became confused and changed her answers, adding a 0 to the dividend instead. In all, 9 out of 15 students were able to use Chang's method successfully on at least one problem, and at least four extrapolated it to apply to multiplication by 100. Also, when asked for an example of a method, 7 children (Aurora, Arnie, Bert, Harold, Josephia, Selvaranee, Yan) described some version of "Chang's method" (7 children mentioned other methods, and Frankie was unable to recall any method.) Finally, Ms. Coleman told us she heard from Bert's fourth-grade teacher that he had brought up this method in class the following year. Clearly, Chang's method made an impression on these students.

Chang Himself at Year's End

Interestingly, on the final interview Chang used his own method on only one problem--600 ÷ 10. For most of the problems, he used either memorized number facts or the conventional multiplication and division algorithms. Chang successfully solved all but one problem on the interview, but was unable to explain the place value significance of the long division algorithm he said he had learned from his mother, or what his answer of 0.5 for 4 ÷ 8 meant; he thought it might be "five-tenths lower than zero," but said "in money it means 5 cents." The word problem involving markers was the only problem Chang failed to solve correctly. He started off well enough, by saying to himself, "26 divided by 4 is 6," but then seemed to get caught up in some kind of checking procedure, multiplying 6 x 4 to get his answer of 24 boxes needed. The interviewer pushed him several times to reconsider this answer, pointing out that there were only 26 children in the class, but he continued to justify his answer by recounting the procedure he had followed: "First of all, I did 26 divided by 4, and I came up with the answer of 6, . . . and then I said 26, I mean 6 groups in the class, and I said, wait . . . (pause--8 seconds) . . . and then I said (pause) um 4 x 6 equals 24, so I think um she'll have to buy 24 markers."

Chang showed little change in his ideas about mathematics or mathematical discourse over the year. Math was still his favorite subject, because "most of the problems [Ms. Coleman] gives us is pretty easy . . . the part of math I like best is when she gives us workbook pages or sheets of paper that have math problems on it." He sometimes liked Mad Minutes, but "I don't like the problems that trick me, like when we have been doing times, a lot of times, when she changes instantly to plus." He said that Yan was the best student in math, "because in Mad Minutes we always race for the first one to finish. And then sometimes, I passed Yan about four or five times, and he passed me the rest of the times, but he's better than me." People who are good at math, "not only they can get it fast, they, you know, they get it right." He said that he and Yan had gotten good at math because,

> In China, they give us math homework, and we have to do it and do it and do it just, and if you always do the math and get not only homework, then your mother gives you some math and doo da doo da (gestures to show repetition), finally, you just get good at it.

He described a method as "when someone comes up with a statement that um the teacher wants to write down on the board . . . she'll just put it down because it's something that a student said that was important, and she knew that was true." He noticed that the classroom discourse was different from last year's, but said he doesn't mind being challenged by other students: "I don't feel sad at all. I just feel that they are kind of learning. I feel that they're brave to comment on other people." When someone like Yan disagrees with him, "I just raise my hand and comment on what he says, and at the end we'll find out who's true and who's not."

In the small-group problem-solving session, Chang and the other two members of his group had a great deal of trouble cooperating. Although the interviewer stated clearly that their task was to "work on it with your group and come up with an answer you agree on," each member of the group began to work independently to solve the three word problems. Because there was not enough play money to model three sets of problems, Chang and Melissa began squabbling over the money:

> Chang: Melissa, you are taking all the quarters. Do we have ten dollars?
>
> Melissa: (pause, counting her coins) No.
>
> Chang: Ah, boguuus. . . . I get the tens, you have the fives and ones.
>
> Melissa: I don't care, I have all the quarters, almost.
>
> Chang: (Sarcastically) That is very funny, Melissa (paused as Melissa counted her money) You are weird, Melissa.

After reaching independent solutions, they began to discuss which problems they thought were most similar. When Melissa asked Chang why he thought A and C were most alike, he just said, "B and C, A and C, I don't know, I am just going with my answer, A and

C beats B and A." Melissa then tried to draw Josephia, the third group member who had said nothing at all, into the conversation, but Josephia did not respond. Melissa and Chang continued to work independently for a while, and then Chang started to lay out money for Josephia, who sat there rather passively. Melissa tried to get Chang's attention while he was counting out Josephia's money, and Chang said, "Excuuuse *moi*, I am *teaching* you something." Melissa and Chang engaged in parallel monologues for several minutes, until Chang called out, "Yay! we did it, we did it; I am done!" Melissa moved over to see what he had written on his paper, and Chang shielded it, saying "Excuuse *moi*! Do not copy *moi*!" Melissa giggled, and Chang continued, "I hate you!" At this point, the interviewer reminded them that they were supposed to be *agreeing* on an answer, and Chang covered the paper he had been working on, saying, "I am not going to let her copy and she is copying *moi*!" He began to place coins carefully on his paper, saying, "I am going to cover each word I wrote with nickels . . . "

Understanding Chang

Over the course of the year, Chang did not change his understanding of what mathematics is and how one goes about doing mathematics, even within the context of Ms. Coleman's new discourse-based methods. Rather, he filtered through his own lenses what happened during mathematics class, and he understood the purposes of the classroom discourse from the standpoint of his own well-developed and strongly held views.

Chang continued to view mathematics as a competitive process of trying to get right answers as quickly as possible. He had learned the procedures for doing mathematics quickly, mostly from working at home with his parents, but did not seem to value or develop a real conceptual understanding of what he was doing. As a result, Chang relied heavily on following the procedures, as, for example, when he argued so strongly with Ms. Coleman that you had to "move the 1 over . . . because in [multiplication] you're supposed to go from the right side to the left side." Toward the end of the year, Ms.

Coleman speculated that Chang "probably has not really taken problems apart and looked at them to really figure out how they work." Chang sometimes got answers using correct procedures, but then interpreted them incorrectly, as he did with his answer of "0.5" for "4 + 8." He also sometimes got caught up in an incorrect process--rotely applying a set procedure without regard for the "sense" of the answer he was getting; this seemed to be what happened to Chang on the "marker" problem.

Although Chang got many of his ideas about mathematics from his parents, he was able to retain these views because they "fit" some of the activities that were sanctioned in Ms. Coleman's mathematics class. For example, Mad Minutes was a significant activity to Chang. Mad Minutes was the only activity on which the children reported publicly on their scores and then were graded, and performing well on Mad Minutes required speed and accuracy, rather than understanding and reflection. Ms. Coleman herself sometimes focused on efficiency of explanation, rather than on pursuit of understanding the underlying ideas. For example, when Arnie recognized that Chang's "add-a-0" method could be generalized to include multiplying 10 x 0, Ms. Coleman did not see the significance of his argument, and rejected it. Indeed, during the whole discussion of Chang's method, Ms. Coleman never asked *why* Chang's method worked. Although the children brought up place value concepts several times in relation to the method, Ms. Coleman did not pursue this connection. Rather, she had students validate the original method empirically by trying "some numbers to see if it work[ed]." This emphasis on finding ways that "worked" to do problems was a main focus of Ms. Coleman's teaching that year, and it was related to her use of the word "method" to describe the student's ideas, which differed from the words "conjecture" or "idea" that were often used in Deborah Ba[illegible]s mathematics class.

A continued emphasis on mathematical procedures rather than conceptual understanding might handicap Chang later, in spite of his precocity in math thus far. Ms. Coleman drew the following insightful conclusion about Chang:

> He's got a lot in his head because he's been taught things by his mom, by his dad, or whomever. . . but he's kind of hung on to bits and pieces . . . I think probably for most of these things he does--there's some holes in his thinking.

Chang's efficiency in computation will probably help him continue to be successful in mathematics throughout elementary school. The challenge for Chang may come later, in courses like geometry, in which success is based not on memorizable procedures, but more on an understanding of and willingness to investigate a problem space, to formulate and test a variety of hypotheses in much the same way that Ms. Coleman hoped her students would learn to do through discourse in her class. Chang did not really "understand" that discourse, and that way of thinking, in part because he was often sure he already knew. He saw his role in the classroom discourse as one of "teaching" or telling his classmates things he already knew. For Chang's classmate, Ataia, it was a different story--she struggled with the classroom discourse precisely because she was never sure when or if she "knew."

Ataia's Story: When Is knowing?

Ms. Coleman reported mixed impressions of Ataia during the months before the multiplication and division unit. In January, she said:

> I feel that Marta, Tessa, and Ataia are my real thinkers in here in terms of math . . . those three girls are really just kind of coming out and not afraid to take a risk or not afraid to share . . . They just seem to be thinking about my questions a little bit more in depth than some of the others.

Yet, in March, when Ms. Coleman was explaining how depressed she had recently been by some students' explanations during class discussion, she said:

> Calvin, who I knew was really kind of weak also, and Ataia, I guess she just kind of confirmed what I've already thought. Even though she is willing to take a risk and try to explain something, her explanation just gets lost somehow, and it doesn't really make sense, and she isn't able to pull things together . . . so those are the kids I saw yesterday, . . . and I thought, "Oh, my God!"

Knowing, Sometimes . . .

We, too, got rather mixed impressions of Ataia when we watched her during the multiplication and division unit. Ataia was one of the quieter students, not talking at all during 3 of the 18 class discussion days in the unit, and often speaking only once or twice on the days that she did talk. None of the methods posted or discussed during the unit was labeled "Ataia's method." Ataia did make substantive contributions to the discussion on 6 days, however, which was about average, and many of her remarks seemed to show a reasonable understanding of the particular topic or method under discussion.

For example, on the first day of the unit when Chang introduced his add-a-0 method for multiplying by 10, Ataia reexplained it clearly, saying "Chang said that if you have 10 times, like, 8, and it equals 80 (writing 10 x 8 = 80 on the board; this is the example Chang had used), all you have to do is add a 0 to it (she adds a 0 after the factor 8)." The incident that followed, however, showed that Ataia may have been less sure than she sounded. Ms. Coleman urged her to give a new example, and Ataia at first said she couldn't, but then began to write 10 x 2 = 20 on the board. Another student, Arnie, gave a small giggle (as he later explained, because he was "thinking of a huge number"), and Ataia immediately erased the problem. Only after Ms. Coleman took Arnie to task and reassured her did Ataia resume her explanation, showing that placing a 0 after the 2 would give the answer of

Two days later when the class was trying to figure out how to read the notation $10\overline{)350}$, Yan and some others were saying it is "10 divided by 350" while most said it was "350 divided by 10." Ataia proposed that "350 divided by 10 is the same as 10 divided by 350," perhaps echoing the commutivity of multiplication idea which the class had discussed earlier. During the next session, Yan "revised," saying that "you can't do 10 divided by 350." Most students agreed with him, but Ataia still disagreed, saying she thought you could do it, but she was unable to explain how.

Several days later, Yan demonstrated a "check" for 417 ÷ 10 = 41 r7, which involved multiplying 41 x 10 and adding 7. This seemed to be a new and difficult idea for most of the students, and a rather confused discussion of the relationship followed, during which Ataia pointed out that the 41 in the multiplication problem was the same number as the answer to the division problem. Yan decided to give a simpler example, and demonstrated how 40 ÷ 5 would equal 8, by subtracting a series of eight 5s from 40 until he had nothing left. He then demonstrated the same problem (40 ÷ 5) using base-10 blocks. Arnie then proposed that you could add 5s until you got to 40, and Aurora suggested that you could "do it backwards," like ☐ x 5 = 40. Ataia did not contribute to the discussion of this method, but carefully carefully copied it, with the examples from the board, in her notebook.

When Chang proposed his "cross-off-a-0" method of dividing by 10, giving as an example that 450 ÷ 10 = 45, Calvin disagreed with him. Ataia supported Chang, drawing an important connection between "when Chang said if you multiply the number by 10, just add 0, but I put away the 0 . . . if you cross off 0 to find the answer, then you can add that up and get 450."

On April 25, Ms. Coleman interviewed Ataia about some homework on which she had solved $8\overline{)40}$, $8\overline{)400}$, 60 ÷ 10, 600 ÷ 10 correctly, using "Aurora's method." However, Ataia gave only whole number quotients for 213 ÷ 5 and 234 ÷ 30, ignoring the remainders. She gave 5 for the quotient of 44 ÷ 9, saying "45 is closer to 44 than 36 is." She successfully solved 65 ÷ 5 by counting by 5s, a strategy she also mentioned in reference to $8\overline{)40}$. Although Ataia did not claim to have noticed the connection between $8\overline{)40}$ and $8\overline{)400}$ while solving them, afterwards she said that she could tell these answers were right, because of something about "the 0s" (the tape was nearly inaudible at this point). In an interview later with us, Ms. Coleman expressed concern about Ataia's idea that you pick the "nearest" number, as she did in 44 ÷ 9, which, according to Ms. Coleman, her mother had told her to do.

In early May, during a discussion of whether you should count by the first number in a multiplication problem, specifically whether you should count by 10s in doing 10 x 4, Ataia suggested that counting by 4s would give you the same answer, but "it would be longer." When this topic recurred on May 4, Ataia agreed with Chang that using the larger number to count by would be easier. On the same day, Marta demonstrated her solution for 40 ÷ 12 by drawing an array of 40 dots. After getting off to a false start by circling two groups of 12 and two groups of 8, Marta solved the problem by circling three groups of 12 dots each, with 3 left over, yielding an answer of 3 r4. Ataia broke in at this point, telling Ms. Coleman that Marta did not start with 40 dots. Ms. Coleman replied that it didn't matter, "I'm not necessarily going to pick apart her picture because she didn't have 40 [dots]." To us, however, Ataia did not seem to be making a criticism, but rather voicing a concern that, if Marta did not start with 40 dots, her circling-groups-of-12 procedure would lead to an incorrect answer. In her notebook for that day, Ataia had written

$\begin{array}{r} \underline{12} \\ 12\overline{)40} \end{array}$ and $\begin{array}{r} \underline{4} \\ 12\overline{)40} \end{array}$ along with a diagram of 40 dots,

with 10 groups of four dots each circled, showing that she remained somewhat confused about what Marta had done.

On May 7, Ataia had solved 10 multiplication and division problems in her notebook, and indicated that she had used serial subtraction or addition to either solve or check (it is unclear which) all of them. Included were 9 x 6 = 54 and 56 ÷ 8 = 7. The next day the class continued to discuss the relationships between multiplication and division. Ms. Coleman put up on the board three problems from the previous day: 4 x 8 = 32, 32 ÷ 8 = 4, and 32 ÷ 4 = 8, and asked if anyone saw a relationship between them. Josephia noted the "numbers are the same." Ms. Coleman asked for a more complete explanation, and Ataia pointed out the various positions each number takes in each problem, but could only repeat that "they are all the same." When Ms. Coleman urged the students to talk about more than just these specific numbers, Tessa suggested that a different set of numbers "would do the same thing," and Ataia elaborated this by writing "2 x 4 = 8," "8 ÷ 4 = 2," and "8 ÷ 2 = 4" on the board. She explained that her example was like Calvin's example on his "method," which was posted above the board and stated that "division is just like times" giving the example that 40 ÷ 8 = 5 and 8 x 5 = 40.

Finally, during the last day of the unit, when Ms. Coleman was trying to wrap things up, she asked the students what they would do to solve 102 ÷ 8. Ataia suggested that she could "count by 8s" in a questioning tone of voice which led Ms. Coleman to say, "You're asking me? Tell your classmates."

In a brief in-class interview with Ms. Coleman at the end of May, Ataia solved 28 ÷ 7, saying she "tried" 7 x 1, 7 x 2, 7 x 3, and then 7 x 4 until she got to 28. She also solved 6 x 7, saying she "just wrote it down, 7 six times."

Ataia at Year's End

Given the reasonable amount of understanding Ataia seemed to show during class discussions, we were surprised at how hard a time Ataia had with the multiplication and division problems we posed on the individual interview in June. Ataia was able to solve only 4 of the 12 computation problems, and although she attempted to use 7 different strategies, she was only able to use 4 strategies successfully. For example, she tried to use Chang's method to solve 60 ÷ 10 and 600 ÷ 10, but got confused about whether she should add or cross out a 0, finally adding a 0 to each dividend, getting answers of 600 and 6000 respectively. She was unable to solve 6 x 9, losing track while attempting to count off six groups of nine on her fingers, and she wouldn't even try 56 ÷ 8, even though she had used serial addition to solve both problems just a month earlier in her mathematics notebook. Ataia started off well enough on the marker problem, drawing 26 dots and then circling six groups of 4, but then she decided to "times 26 four times," wrote four 26s in a column and added them to get her first answer of 114. Then she added 26 + 14, and gave her final answer as 40. When the interviewer asked why she had not used her drawing of the dots, Ataia replied, "I don't think you should have a remainder in addition." She also had trouble with the problem 50 - 14, at first "bringing down" the 4 and not regrouping, and getting an answer of 44. Then, when the interviewer asked her if she remembered anything about "borrowing," she redid the problem correctly to get 36 using the regrouping algorithm but was unable to decide which answer was correct.

Ataia's other answers during the interview reflected a lack of self-confidence that matched her poor performance in math. Although math was now her second favorite subject, she said she didn't "like math that much." Division was hard because, "I get times and division mixed up." When asked whether people ever disagreed with each other in class, she said it "happens a lot of times" and "they yell at each other until someone says the right answers." When asked about herself, she said "a lot of people" had disagreed with her

> lots of times . . . It's unfair! Because they agree with other people. and they won't even let me have a chance to say it! Like if I just say one word, that's what they think I mean. So it, um, so they, um disagree with me.

When asked what she thought they should do, she said, "Maybe sit and listen until I'm through. Maybe [then] agree or disagree." She recalled that one time she

> had a method, but Ms Coleman thought it was a little too long. And there were three people, and one was me and was Chang and one was Harold. We had to find out a strategy for Marta, and mine didn't say much, and it was long too, so she picked, um, I think, Chang. . . . I didn't mind . . . they're smart (smiling).

Another time, "[Ms. Coleman] had a problem on the board. We were trying to find out the pattern. I finally, um, thought it, but it was too late to raise my hand."

Chang, too, had fallen in her estimation. Ataia shared with the interviewer, "Um, um, there's a boy in our class. Um, his name is Chang. I don't think I should be telling you this. But he writes things on the boards and, um, sometimes it's wrong and Yan comes up and corrects it all the time." She guessed that Yan may have gotten so good in math because of his parents, "Maybe [his mother] was a math teacher or something. Like maybe Yan's dad is a math teacher."

Ataia worked with two other girls, Marta and Selvaranee, during the small-group problem-solving session in June. In the beginning, Ataia seemed to understand the word problems, and, in consultation with the others, arrived without much trouble at the correct solutions. She was even able to help Marta understand why Marta's initial answer to one problem was incorrect. But when the discussion shifted to the questions of which problems were most similar or dissimilar, questions to which (unsuspected by most of the students) there were no "right" answers, Ataia seemed to become less comfortable. She tried to explain her own idea that problems A and C were most alike, but the other two girls did not agree with her, and her explanations grew more intense and less coherent. Then she rather suddenly withdrew from the discussion, writing on her paper whatever the other two suggested, and only shrugging silently when the interviewer asked whether she really agreed with their conclusions.

Understanding Ataia

Ataia's case is the one that has puzzled us the most. In an interview at year's end, Ms. Coleman described Ataia as "kinda on and . . . kinda off. It depends. You know, she's off, then she's on, and I'm not quite sure why that is." In our fieldnotes, we noted that Ataia's knowledge seemed particularly "fragile." At certain times or in certain situations, Ataia seemed to "know" something, and then at other times or in other situations, she seemed not to "know" it. For example, Ataia discussed and even explicated Chang's method in class very clearly, yet she appeared confused when trying to use it during the final interview. Similarly, Ataia's comments in class seemed to show a pretty good understanding of both the relationship and the difference between multiplication and division, yet Ataia herself said she tended to get them "mixed up." This "mixed" understanding was evident in her use of the multiplication version of Chang's method on two division problems and her multiple attempts on the "marker" problem. What should we make of Ataia's seeming inconsistency and multiple attempts? From one perspective, they would indicate confusion or misunderstanding; from another perspective they make her seem to lack confidence; from yet a third perspective this same "inconsistency" and these same attempts might be viewed as just part of the uncertain process of "making sense."

Perspectives on Ataia: Misunderstanding, Lacking Confidence, or Making Sense

Ataia's understanding of multiplication and division seemed to shift with the context. For example, she seemed to understand better in the same-sex, small-group interview situation, where she could get clues, immediate feedback, and support from the other students, than she did on her individual final interview. Similarly, many of Ataia's comments during whole-class discussion involved follow-ups, expansions, or agreement with other's ideas. Yet the understanding that Ataia demonstrated in such supportive contexts often disappeared when she had to work independently, or under pressure. It is tempting to say that Ataia tended to give inconsistent or multiple answers in these situations because she did not really understand multiplication and division; that she could follow along in class, but her "misunderstandings" were revealed when she tried to work on her own. Either that, or her learning had been so fragile that she forgot most of what she had "understood" in April by the time we interviewed her in June. Certainly a standard scoring of the problems on the final interview (she got only 4 out of 12 right) would lead to this sort of explanation.

Yet, there is another way to look at the fragile nature of Ataia's knowledge. Holt (1967) suggests that this sort of forgetting may involve lack of self-confidence more than lack of knowledge.

> The children who always forget things in school may not forget so much because their memories are bad, as because they never dare trust their memories. Even when they are right, they still *feel* wrong. they are never willing enough to bet on their hunch that something is so, to turn it into a conviction that it really is so. (p. 98)

Ataia, from her own description, certainly seemed to *feel* wrong much of the time. This lack of confidence in her own knowledge may have hindered her in using, in non-supported situations, the understandings she had developed. Indeed, Ataia entered Ms.

Coleman's class with a reticence that might indicate lack of self-confidence. Ms. Coleman described her demeanor on entry as "always just so unhappy . . . like, you know, nobody loved her . . . she was always very quiet, never smiled or would participate." Her quickness to erase her work on the board, when another student giggled, and her wistful comment that she "didn't mind" when other students were picked, because "they're smart," would seem to support this explanation.

One of Ms. Coleman's goals in experimenting with her teaching was that her students, especially the girls, would participate in learning mathematics and so would learn not to be afraid of math. Ataia did participate in the classroom discourse. Yet for her, at least, the new discourse-based approach sometimes seemed no less threatening than the traditional teaching she had experienced the year before. From her descriptions of discourse events, and her attitude when describing Chang, Ataia seemed to see making a mistake in front of the class as an indication of inability. She had also gotten the idea that she was not quick enough in discussion, so that she felt her comments in class were rarely heard in full and recognized. This idea that she was too "slow" in math may have been reinforced by Ataia's poor performance on Mad Minutes. All of these factors may have contributed to Ataia's feelings of often being wrong and her lack of confidence in her her ideas.

But yet another way of viewing Ataia's inconsistency is as a natural part of trying to "make sense"--a phase which adults and experts often fail to see or remember from their perspectives as more knowledgeable others. Holt (1967) attempted to capture this perspective of the child and the learner when he wrote:

> We are so used to the feeling of knowing what we know, or think we know, that we forget what it is like to learn something new and strange. We tend to divide up the world of facts and ideas into two classes, things we know, and things we don't know, and we assume that any particular fact moves instantly from "unknown" to "known." We forget how unsure we often are of things we have just learned . . . What we must understand is that when a child figures out [something], he does not *know* in the sense that we know, he is not certain, that this is so. For reasons he may not be aware of and certainly could not put into words, he has a flash of insight, a hunch . . . He tries out his hunch, and it works. But because a hunch works this time

> does not mean to a child that he can rely on it next time. In fact, he may not even get the same hunch next time . . . Each time he is right, his hunch becomes stronger and surer; but it takes a long time--longer for some children than others--before it becomes what we think of as certain knowledge. (pp 97-98)

Ataia was not only tentative about whether her "hunches" were right, but she also often seemed to hold multiple hunches in mind at once. Her work on the "marker" problem from the final interview again exemplifies this tendency. She also frequently picked up multiple, often contradictory, ideas from the classroom discourse. For example, Ataia's mathematics notebook entry for May 4 records three possible solutions for $12\overline{)40}$: 12, 4, and an implied solution of 10 (she showed 40 dots with 10 groups of 4 circled). Again, this recording of multiple, and to us, mutually exclusive, solutions can be seen as indicating either uncertainty or misunderstanding of the mathematics and the discourse involved.

Yet this uncertainty and this simultaneous consideration of multiple ideas was in some ways supported by the classroom norms that Ms. Coleman had created. Like her teacher colleague, Deborah Ball, Ms. Coleman was attempting to transfer the authority for knowing to her students. In keeping with that goal, Ms. Coleman resisted telling her students whether a particular mathematical strategy, solution or idea was correct or incorrect. She wanted her students to figure out for themselves whether an idea made sense This goal accords with those of Holt and other constructivists who describe children's mathematics learning as a process of "making sense"; it is also similar to what scientists and researchers do when they consider multiple interpretations or test multiple hypotheses to explain the same phenomenon.[8] Ms. Coleman wanted her students to figure out whether their ideas, strategies, and solutions were "right" or "wrong" within the context of the learning community of the classroom. This is similar to what mathematicians, scientists, and scholars do as they develop knowledge and understanding within the context of their scholarly communities.[9]

---

[8] See Resnick, 1989, for one example of a constructivist description of children's mathematics learning; and Wilson and Wineburg, in press, for one example of researchers holding in mind multiple hypotheses about the same phenomenon and thinking aloud about them.

The question is, was Ataia engaged in doing exactly what Ms. Coleman wanted--making sense and attempting to figure out what is right and wrong within the context of her learning community--or was Ataia simply confused?

On the one hand, Ataia may have perceived Ms. Coleman's emphases on developing and honoring multiple ways of doing problems to mean that all solutions were equally valid. In a traditional math class, Ataia would have been expected to learn by listening and remembering what her teacher said. In this class, Ataia may have taken too literally Ms. Coleman's assertion that her peers could be "teachers." Ataia may have still been simply trying to listen and remember, believing that if she only listened and remembered well enough she would somehow understand the multiple ideas that her peers were "teaching" her. This perspective was inadvertently supported by Ms. Coleman, who continually emphasized that students should listen to others and frequently requested that students repeat what others had said. From this point of view, Ataia seemed not to understand, as Harold did, that some of the solutions and strategies suggested by her peers would be incorrect, and that she would need to think critically about what she heard and winnow out what was not helpful.

On the other hand, Ataia did sometimes attempt to reason things out during discussions. For example, in the discussion from May 4 reported earlier, Ataia did point out that Marta's "circling the dots" strategy for solving $12\overline{)40}$ would not work because she had started with fewer than 40 dots. Ataia's remark might be interpreted as an attempt to reason out and make sense of what strategy would work for that particular problem. Ms. Coleman dismissed Ataia's remark, perhaps because Ataia didn't express her concerns clearly or perhaps because Ms. Coleman interpreted Ataia's remark as a criticism of Marta. As it was, Ataia was left with multiple solutions for the same problem, copied into her notebook, and her uncertainty unresolved.

---

[9] Researchers form scholarly communities of discourse in which they develop shared understandings through published conversations and debate at scholarly meetings and through published writings, and , and most recently through conversations over electronic networks. These communities of inquiry have been called "invisible colleges" by Sir Isaac Newton in the seventeenth century and by contemporary researchers studying the sociology of knowledge (Crane, 1972); schools of thought by Kuhn (1970); and research programs by Lakatos (1976).

For Ataia, more questions remained than answers at the end of the year. For us also, in thinking about Ataia, questions remain: Is Ataia's "lack of understanding" related to her lack of confidence? Is it due to her perception of the goals of classroom discourse? Does it simply reflect a natural part of learning? Or might we need to consider some sort of interweaving of all these explanations in our attempt to understand Ataia? When should a learner's uncertainty be judged as misunderstanding?

Selvaranee's Story--"Sticks" and More Sophisticated Understandings

Like Ataia, Selvaranee was one of the quieter students in Ms. Coleman's class. On 4 of the 18 days where whole-class discussion occurred, Selvaranee did not participate at all, and on many other days, she made only one or two comments. She seemed to pay good attention to the discussion though; she was never called on for not paying attention or chastised in any way for breaking any of the classroom norms. And when Selvaranee did talk, her comments were often to the point and well received, at least by her peers.

Selvaranee made substantive contributions to six of the class discussions, and introduced a number of mathematically important ideas. For example, on April 5 she brought up the idea that if 2 x 2 = 4, then 4 ÷ 2 would equal 2, because it's just "doing it backwards." This seems related to an explanation she had written in her notebook for the "sponge" for that day, that 42 x 10 and 420 ÷ 10 are "the same but opposite." Ms. Coleman did not pursue this reference to the reciprocity of multiplication and division, and the idea did not come up again until April 12, when Aurora used the same "doing it backwards" language in proposing her method for doing division by looking for what she called the "missing addend" (actually missing *factor*) in a multiplication problem.

Selvaranee also anticipated a distinction Ms. Coleman later worked hard to make between "41.7" and "41 *remainder* 7" by suggesting on April 11 that the 7 is a remainder, and refusing to agree with Aurora on April 18 that $91 \div 10 = 9.1$, because, she said, "1 is a remainder." On April 27, when Ms. Coleman asked the students about any "patterns" they had noticed with multiplication and division of 10s and 5s, Selvaranee suggested, "If you divide two numbers that are the same number, it'll equal one." Ms. Coleman replied, "I don't understand what you mean," and instead of picking up on this fundamentally important mathematical concept, she redirected Selvaranee's attention to "these numbers we're talking about [10 and 5]."

A Tale of Sticks

Selvaranee made her biggest contribution to the understanding of her classmates during a discussion of the meaning of division, when she told the class how she had solved $350 \div 10$ by drawing 350 "sticks" (tally marks), and then circling groups of 10. This was the first appearance of what became known as "Selvaranee's method," although in class the week before Bert had argued that $420 \div 10 = 42$ because "if you have 420 dots or lines, and you see how many times you circle 10. I think it would be 42." Bert didn't actually solve the problem using this procedure, though, because he wasn't sure he'd "be able to put that many" dots up on the board. On April 17 Selvaranee's method was formally codified along with several other methods for solving $40 \div 5$; it read "Draw 40 sticks and circle groups of 5."

Selvaranee's method was wholeheartedly adopted by many children in the class, especially those who had trouble with more symbol-dependent methods. It seemed to make intuitive sense to them as both a way of conceptualizing division and a way of performing it, and was referred to and used by students throughout the rest of the unit. Ms. Coleman's attitude toward this method, however, was quite ambivalent. When she talked with us, Ms. Coleman seemed to recognize that this method was intuitively helpful to many students. For example, after individually interviewing most of her students on

April 25 and 26 about their solutions to nine different division problems, ranging from 40 ÷ 8 to 234 ÷ 30, Ms. Coleman commented about Tessa that "if she can draw her sticks and that type of thing, I think she's gonna be fine. I think she understands that," and again that "If they can draw sticks, I think a lot of them would be OK," yet "I would like to see her (Tessa) not have to do that. And the same thing with Aurora, not to have to draw the sticks."

During class discussions, Ms. Coleman's words and actions tended to discourage students' use of "the sticks." As early as mid-April, on the day "Selvaranee's method" was first officially codified and posted, Ms. Coleman indicated that this method was less efficient, and therefore less desirable, than other alternatives. Toward the end of class, Ms. Coleman asked the students to work independently on three division problems, 125 ÷ 10, 740 ÷ 10, and 741 ÷ 11, while she walked around and observed their work. Selvaranee solved the first problem by drawing 125 sticks in her notebook and circling 12 groups of ten, getting a correct answer of 12 r5. After observing Selvaranee and one other girl as they work, Ms. Coleman called the class to attention and said:

> I see someone doing this (she draws a few sticks on the board). OK? What are they doing? They're gonna draw 125 sticks! (class gives a big laugh). I mean, that's quite all right, if that's the only way that you, the only method that you know that you can use in terms of figuring this out. . . . Hold it, boys and girls, because I don't see anyone, really, let's see, perhaps with an answer. When I came around and asked everyone about this (she is referring an earlier problem, 41 x 10), you told me that you used Chang's method. What happened with it? I don't see anyone using Chang's method now [or] Yan's method.

Two days later, when she was going over possible ways to do 471 ÷ 10, Ms. Coleman again suggested that Selvaranee's method worked, "but I don't know if you would really want to do that for every division problem," because it would be hard to draw 471 sticks. Selvaranee suggested that you could "draw one stick and pretend it stands for 100 or for 10." (She had actually done this successfully for three problems on her homework, 91 ÷ 10, 87 ÷ 10, and 123 ÷ 10.) Ms. Coleman did not ask for a fuller explanation of this idea, but added "you can substitute objects to stand for 100" to the bottom of Selvaranee's posted method.

In early May, after another student, Tessa, had demonstrated a way to prove multiplication by grouped addition (i.e., showing that 4 x 8 = 32 by adding 16 + 16 to represent the four 8s), Ms. Coleman endorsed this method enthusiastically, asking the class "Do you need to go through and draw out 32 sticks?" When the class chorused, "Noooo!", Ms. Coleman continued, "So if you were given some multiplication problems, and I said to you, 'Prove each one,' you could either go through and draw 32 sticks, which is going to take you much longer to do . . . or you [could] do some grouping, like this." Following this discussion, Ms. Coleman asked the students to work independently on 40 ÷ 12. After observing several students' work, she said to one student, "We're back to the sticks, Ehhhh?" and then stage-whispered to the observer, with seeming frustration, "They're using the sticks!" At the end of that day's class, after Marta demonstrated her solution by drawing 40 dots and circling groups of 12, Ms. Coleman said, "Boys and girls . . . what I want us to do is become comfortable enough with division . . . so we can get away from drawing the sticks and drawing the dots, OK? That is my goal for you."

During the final week of the unit, however, the "sticks" remained a favorite way for students to "prove" their answers in division, and sometimes to get their answers in the first place. Ms. Coleman again observed Selvaranee working on "proofs" for several division problems, and said, "It will take you forever to do those sticks. Is there some other way you can think of?" Selvaranee replied, "But if I didn't *show* you . . ." and giggled as Ms. Coleman left. The next day Ms. Coleman asked six students to show on the board how they proved their answer for these problems, and four of the six drew sticks and circled them, although Calvin erased his sticks afterwards, and wrote a distorted version of the long division algorithm next to his answer. When Calvin came to explain it, he could not talk through the algorithm, and he explained that he really drew sticks.

When Ms. Coleman asked Calvin to show what he had actually done, he redrew the sticks. Then Ms. Coleman said, "Using the sticks. How many people feel that's the easiest way to prove whether or not your answers are correct? (several hands begin to raise). And if you had to do this in a Mad Minute, do you think you could get it, if you had 30 division problems?" The class chorused, "Noooo!" and Ms. Coleman asked, "What are maybe, some easier ways?"

Finally, a day later Ms. Coleman posed the problem 120 ÷ 8, suggesting that students find "some easier ways" than drawing sticks to do this. She concluded her wrap-up of the whole unit by saying, "I think in my own mind, boys and girls, just trying to group numbers together might be easier. Some of you might want to go through and do this (the sticks), although you know it takes much longer. But it works, OK?" She then asked if the recent CTBS test they had taken allowed them time enough to draw sticks.

By the time we did final interviews with students in June, Ms. Coleman's strategy of discouraging the use of "sticks" seemed to have taken effect. Selvaranee's method was used by students on only 3 of the 12 computational problems we posed, and then by only a few students. However, students' movement away from using sticks had questionable effects. First, some students simply shifted to another form of direct representation, for example, substituting the white "unit" blocks or their fingers for the sticks, but in all other ways counting and grouping as before. Second, several students were unable to solve some of the division problems that they attempted with more sophisticated strategies, problems whose type they had solved successfully in class by using sticks or some other form of direct representation. For example, in June, Aurora was unable to do 56 ÷ 8, 230 ÷ 23, or 132 ÷ 11, the last two of which she attempted using the long division algorithm. Yet in April, during in-class interviews by Ms. Coleman, Aurora was able to solve and explain three problems of similar difficulty, 400 ÷ 8, 72 ÷ 4, and

234 ÷ 30, using Selvaranee's method. "Get[ting] away from drawing the sticks" did not seem to have been beneficial for all students.

Selvaranee at Year's End

By year's end, Selvaranee herself seemed to have moved successfully beyond using her sticks. She was able to do 10 of the 12 multiplication and division computation problems we posed, and only Chang, Yan and Bert did better than this on the year-end interview. Further, Selvaranee used a wide variety of strategies, although she still used a form of direct representation with base-10 blocks on the more difficult division problems. She also demonstrated a sound conceptual understanding of what she was doing. She recognized the reciprocal relationship of multiplication and division, and used it derive the answer for 60 ÷ 10 from 6 x 10 = 60. She was one of the few students who noticed that 230 ÷ 23 was related to the other "0" problems, specifically volunteering that it was like 60 ÷ 10, 600 ÷ 10, 8 x 100, and 6 x 10, because they're "not the same numbers, but they're related because they do the same thing . . . add a 0 [or] add two 0s like the one up here (8 x 100)." She also solved the "marker" word problem by remembering 6 x 4 = 24, although she could not do this problem at the beginning of the year even by drawing a representation or using manipulatives. Selvaranee was equally successful on the addition and subtraction portion of the interview, easily solving two number problems that had defeated her during the initial interview.

Selvaranee also demonstrated considerable abilities in mathematics and facility with mathematical discourse during the group problem-solving interview in June. She solved all three word problems on her first attempt and took a leadership role in her group of three girls: advancing solutions, justifying her thinking, and holding to her views, but also making sure that each person was heard, frequently asking things like, "And what did you think, Ataia?" and not moving on until a group consensus and understanding had been reached on each problem.

Selvaranee's success was somewhat surprising to us; we had not expected her to do so well. Our initial picture of her from our classroom observations was of a rather average student with no great mathematical understanding, yet our final data did not go with this picture (and Selvaranee's CTBS overall math score, obtained later, reinforced this final data--she scored in the 82nd percentile nationally, the fourth highest in Ms. Coleman's class). As we searched for the reasons for this discrepancy, we noticed several factors which may have distorted our initial picture. First, as previously mentioned, Selvaranee, though by no means silent in class, was one of the quieter students. As both teachers and researchers often do, we may have mistaken reticence for ignorance. In a class like Ms. Coleman's, where so much depended upon whole-group discussion, this may happen particularly to girls, who tend to be less assertive in their participation in classroom discussions (Wilkinson & Marrett, 1985). In analyzing our own classroom observations, we found that *all* of the boys participated in class discussion at or *above* the median number of days during the unit, while *all* of the girls participated at or *below* the median number of days--the split was perfect. Selvaranee's much more vocal behavior in the small-group problem-solving experience with Ataia and Marta might indicate that gender was particularly influential in her case.

Second, until we analyzed our videotapes, we had not realized the number of insightful mathematical ideas Selvaranee had proposed during discussion, perhaps because these ideas were rarely followed up. Finally, we found that our initial ideas about Selvaranee had been reinforced by the emphasis placed in class on faster, more efficient ways of doing math. Ms. Coleman herself did not rate Selvaranee's understanding as particularly high, feeling that her preference for using "the sticks" showed an immaturity in conceptualization. Selvaranee also did only about average (B/B-) on the Mad Minutes timed tests, which were the only regularly graded tasks in mathematics. Perhaps because these results were publicly collected, or perhaps because they associated her with "the sticks," none of the other students named Selvaranee as someone who was particularly good at math.

Unfortunately, there is some evidence that, by June, Selvaranee had begun to share these opinions. Although she was willing to try every problem on the interview, and showed a lot of persistence, especially in solving the long division problems, when she saw that she has mistakenly done 4 x 12 as a division problem, she was not willing to go back and revise it, although urged to do so. She retained her traditional ideas about what "being good" in math meant; she still chose Yan as the classmate who was good at math, and still because "he knows his times tables up to 100 or something" and "never missed" on Mad Minutes. In contrast to her earlier positive response, when asked in June whether she liked math, Selvaranee replied, "Sometimes I do, but sometimes I don't. I don't like it when we do too hard things and we have to answer hard questions . . . I like it when we do easy things and play games with math." She said that when someone disagreed with her in class, "It doesn't feel good, but I think some of them are right that they disagree with me, so I don't really care," but when people agree with her, it feels good, "because you know you got it right, maybe you know you got it right." If there were no one around to agree or disagree, the only way she could think of in June to see if a solution was correct was to "check it . . . like do it over and see if you were right."

Understanding Selvaranee

What happened to Selvaranee in this mathematics class and why? On the one hand, Selvaranee did some important learning. She gained in her understanding of and her ability to solve problems in addition, subtraction, multiplication and division, and she developed a number of different, flexible strategies in these areas. Many of these strategies can be traced back to methods suggested by other students, then discussed, refined, and used in class. Selvaranee's behavior during the small-group problem solving session showed that she also has developed some powerful tools for mathematical discourse: ways to propose, defend, and evaluate mathematical ideas while maintaining group cohesiveness and furthering group understanding. Those were the same tools that Ms. Coleman originally had hoped to develop in her students by adopting this new discourse-based type of teaching, and in Selvaranee's case these hopes were realized. In the class discussions during the multiplication and division unit, however, Selvaranee's mastery of these tools was not as much in evidence.

On the other hand, although she had gained substantially in her mathematical understanding, by the end of the year Selvaranee had lost some of the self-confidence and enjoyment of mathematics that she had expressed at the beginning of the year. Part of this loss might be traced to happenings during mathematics class. Although Selvaranee's method was formally sanctioned by Ms. Coleman and adopted by many students, informally Selvaranee and other students were pressed to "get away from drawing the sticks." This pressure may have led Selvaranee to feel that her way of doing and understanding division was somehow less "advanced" or less acceptable than others'. During the classroom discourse, Selvaranee's method was singled out as slower than other methods, and Selvaranee's use of her own method may have handicapped her performance on Mad Minutes. Although Ms. Coleman did not intend to emphasize speed of calculation as a goal in her class, throughout the year students' performance on Mad Minutes continued to be used as a gauge of mathematical ability by Ms. Coleman, the students, and Selvaranee herself. This may have been due to the daily public disclosure of students' scores and the fact that letter grades were assigned only to this activity, and none other. Selvaranee may have experienced some frustration and felt less good at math when she was unable to complete many problems and score very well on these timed tests.

Understanding Learners' Understandings

These cases of five learners revealed how, within the same mathematics class, different students had different experiences and constructed different understandings both about themselves as learners and about the mathematics they were supposed to learn in Ms. Coleman's new discourse-based mathematics class.

Two students, Harold and Selvaranee, came to understand the purpose of mathematical discussion to be creating mathematical knowledge for themselves and others. They participated significantly in the classroom discourse, navigated through it with fluency and flexibility, and developed sound mathematical understandings through their participation. Another student, Chang, understood the purposes of classroom discourse quite differently from Harold and Selvaranee. Most of the time, he was attempting to tell or transmit what he knew to other students, usually believing he knew the mathematics that others needed to learn. Chang performed well in discussion and on computation problems in the final interview because of prior algorithmic knowledge that he had been taught by his parents. Yet he failed to grasp the essential concepts underlying the procedures at which he was so adept, and his participation in the discourse did not lead him to develop greater conceptual understanding.

Both Chang and the two girls we have studied, Ataia and Selvaranee, revealed that self-confidence was a major issue related to success in this kind of discourse-based "teaching for understanding." In spite of some loss of self-confidence, Selvaranee was able to persist in discussion and use her sticks to ground her mathematical understanding in the face of pressure to move to more efficient methods. Ataia, who came in lacking confidence, and Chang, who seemed overconfident, devewloped very differently in classroom discourse. Chang maintained his rather rigid, procedural approach to mathematics in the face of others who developed more conceptual ways of doing problems because he knew he was "right," and he thought his role was to "teach" his classmates. Ataia, on the other hand, ended up feeling "dumb" and seeming to have a fragile understanding that crumbled easily under pressure. When Ataia's understanding was supported, as in her small-group work with Selvaranee, she showed she could solve difficult word problems and even help her colleague, Marta, understand why her answer was incorrect.

Finally, there was Calvin, who continued to see school mathematics as coming up with right answers, even though his teacher saw mathematics as more than that. Calvin refused to "buy into" class discussion because to him it seemed a waste of time; he thought it would be much quicker just to be told the right answer by the teacher so he could learn it. Calvin spent much of his time in a world separate from school, a world of his own--"dreamyland". This world for Calvin was a world of the "authentic"--things he cared about and things that interested him. What it meant to "know" mathematics in Calvin's "authentic" world seemed very different from what it meant to "know" in the world of school mathematics according to Calvin and what it meant to "know" mathematics in the world of his teacher and other students such as Harold. Rifts between these different worlds opened up and gaping holes became only too apparent in the discourse between Calvin and Ms. Coleman and between Calvin and his peers. At the end of the year, these gaps for Calvin still remained and were not bridged, but perhaps even widened, by his move to another school the following year.

The stories of these five learners highlight the importance of discourse, knowledge, and student roles in the development of learners' understanding. These three--discourse, knowledge, and roles--were interwoven in diverse ways that helped to form the fabric of students' understanding. How students understood the purposes of the discourse, what they thought it meant to know mathematics, and the roles with regard to authority that they thought they were to assume--all these were related to the mathematical understandings that students came to develop in Ms. Coleman's classroom.

Purposes of Classroom Discourse

In some important ways, the discourse in Ms. Coleman's classroom was typical of the discourse in most mathematics classrooms . First, the pattern of discourse was teacher-student, teacher-student, teacher-student. On only a few occasions did we observe a pattern of student-student discourse develop, and then for only a brief period of time, because the teacher intervened. Second, the teacher continued to be automatically accorded the floor, no matter who was talking. Third, even though Ms. Coleman wanted class discussions to be opportunities for learning, in which students could try out new ideas, build arguments, and revise their thinking, some students still saw these discussions as traditional classroom "recitations," occasions for being right or wrong, for showing up their ability or their ignorance in mathematics.

Yet in other important ways, the discourse in Ms. Coleman's mathematics class was also remarkably different from the discourse that has characterized the typical elementary school mathematics classroom. First, questions that were posed were divergent rather than convergent. It was expected that there would be a variety of acceptable strategies for solving the mathematical problems that were posed. Second, students, rather than the teacher, decided on the correctness or incorrectness of the solutions that were offered. Students' opinions were solicited. The teacher asked them whether they agreed or disagreed with the solutions or strategies that had been offered, and she required that the students explain what they agreed or disagreed with and why. Third, the amount of student talk was considerably greater than in a traditional classroom where teacher talk typically prevails. Students were expected to talk and to participate in the class discussion; they were told that they would be called upon to participate and that they should be listening to their peers at all times, so if they were called upon by the teacher, they could explain what other students had said. Finally, although the problems were posed and situations set by the teacher, the content discussed came from the students--students' ideas and methods formed the "stuff" that was discussed by the class and potentially, the mathematical knowledge that was to be learned.

Just as Ms. Coleman had readily learned to initiate and manage this new kind of classroom discourse from watching Deborah Ball, Ms. Coleman's students readily learned to participate in this discourse that differed in many ways from what they were accustomed to in their earlier educational experiences.[10] They quickly learned to "talk the talk" and "walk the walk," but, in the very same class, different students seemed to develop quite different understandings of why they were engaging in this new form of discourse. Would more explicit discussion about the purposes and norms for discourse in their mathematics class have helped students like Chang, Ataia, and Calvin construct the kinds of understandings that Harold and Selvaranee developed? In a similar situation, Paul Cobb and his colleagues (Cobb, Yackel, & Wood, 1989; Cobb, Wood, & Yackel, 1991) have reported such explicit discussion to be helpful for constructing a shared understanding of norms and purposes for discourse within the mathematical community of a classroom.

Listening to demonstrate attention to your teacher. Such discussions occurred only rarely in Ms. Coleman's class but when they did, the discussions opened real windows into Ms. Coleman's and her students' different understandings of the purposes for the classroom discourse. A discussion on April 18 (mentioned earlier in our case study of Calvin) provided such a window. On this day Ms. Coleman initiated a discussion of why the students should "listen" to each other when they talked during mathematics class. One student suggested that they need to listen so that if the teacher calls on them, they will know what was said. When Chang and Melissa repeated this same idea, Ms. Coleman responded by saying that she wanted a more "positive" reason for listening. Frankie then started to say something about "If you don't know it, and you call on them," but he was interrupted by Ms. Coleman who stopped him, saying that Frankie was the "fourth person" to say this, and she wanted something more "positive." This was a puzzling discussion for us and perhaps for the students because in continuing to reiterate their idea about why they were to listen, students were restating words they had heard their teacher herself say often during mathematics class. For example, only the week before Ms. Coleman had told the class:

---

[10] Assumptions underlying Deborah Ball's classroom practice have been described by Ball herself (Ball, 1993) as well by Peterson and Knapp (1993). Ball and Coleman differ in their assumptions about mathematical knowledge and about how students come to know and understand mathematics. Watching Deborah Ball teach for a week gave Keisha Coleman many ideas about new ways to teach math, but she had little chance to talk with Deborah, or anyone else, about *why* Deborah taught the way she did or about Deborah's ideas underlying her discourse-based approach to teaching for understanding. As a result, Ms. Coleman knew, for instance, that she wanted kids to talk more in class, but she still saw this talk as revolving around procedures, "methods" for doing problems, rather than around more conceptually-based understandings of mathematics and our number system. For example, during all the classroom conversation around "Chang's method," Ms. Coleman never asked the students *why* Chang's method might work. Such a question might have enabled students to consciously explore the base-10 nature of our number system, and provided them with additional opportunities to develop their understandings of place value, a concept which Ms. Coleman herself characterized as one in which many of her students were "not very strong".

> You must know that I call on you when I don't think you're paying attention a lot of times. . . . You need to make sure that you are fully aware of what people are saying so you can repeat it. If you repeat it, then that at least lets me know that you were listening to what your classmates were saying.

Listening to students as teachers. However, during the same lesson, Ms. Coleman had also reiterated a second purpose for listening--that students would learn from each other. In this case, Ms. Coleman called on Marta to explain with the express purpose that other students would learn from her. She directed the students as follows:

> I want you to be able to comment. She's [Marta's] the teacher right now so let's see how well she can teach you what she thinks she knows about this particular problem. And let's see how well you--you will be *as students* learning.

This second purpose for listening was the more "positive" reason that Ms. Coleman was seeking from students during their discussion on April 18. When students didn't mention this reason, Ms. Coleman finally told then what she saw as the purposes for the discussion:

> We have several strategies on the board, and these are strategies that you folks came up with. I didn't give you any of that information, did I? (The class replied, "NO!" in unison.) You have become very good teachers of your own thinking. And, perhaps if one of these strategies doesn't work for you, maybe you can determine a strategy that works--something that might help someone else. That's kind of what we're trying to do--make sure that we can come up with ways that we can figure things out.

This purpose for classroom discourse was voiced several times in subsequent weeks as Ms. Coleman often called on students to explain or teach other students. Some students did internalize this ideas of themselves as teachers. For example, on the last day of the unit, when Ms. Coleman asked the students, "What have I taught you about multiplication?", Harold reframed the question and suggested that he didn't think that Ms. Coleman had taught them at all because she wasn't "the person that came up with the methods . . . other people in the class did." Arnie agreed, "We came up with our own methods," and added, "We were the ones who had to prove them." He concluded, "I don't think you taught us at all."

The Roles of Teachers and Students

The two purposes for listening to the classroom discourse existed side by side, yet suggested different roles for Ms. Coleman as the "teacher" in each case. By calling on students to ensure that they were paying attention, the teacher maintained authority in discipline and classroom management. But by calling on students to become mathematics teachers themselves, Ms. Coleman was attempting to transfer control for mathematics learning to the students. These different purposes suggested to us a possible tension between Ms. Coleman's attempts to relinquish the role of "final authority" in mathematics and yet retain and exercise authority in classroom management and discipline. For example, classroom management strategies such as the Assertive Discipline techniques seemed to run counter, in many ways, to the classroom norms and relationships that Ms. Coleman was trying to establish in teaching mathematics for understanding. These issues led us to think more deeply about how teaching for understanding makes problematic traditional teacher and student roles, not only in academic discourse, but in all aspects of classroom relationships.

In thinking about Calvin's case, we found that we could not easily separate the effects of classroom discipline from the effects of classroom teaching--the teacher's relationship with Calvin, and thus, in part, Calvin's relationship to the mathematics his teacher was trying to teach him, seemed to be influenced by all the teacher's interactions with him. In Calvin's case, every time his teacher disciplined him for inappropriate behavior or attempted to compel him to pay attention to class discussions, Calvin seemed to withdraw into his own world and to pull further away from participation in the classroom discussion of mathematics. Yet as a responsible public school teacher, Ms. Coleman probably felt that she did not have the luxury of simply waiting until Calvin was "internally motivated" to study multiplication and division.

Because of the interrelated nature of the relationships among the personal, the social, and the "academic," we see classroom management and student motivation as inextricably intertwined with classroom learning of mathematics. Consideration of these interrelations is critical to issues of coherence in purposes and goals for mathematics learnings, coherence in beliefs about knowing and learning that students develop, and ultimately, to students' development of mathematical knowledge and understanding.

A second area of tension became apparent as Ms. Coleman struggled with her own roles and responsibility as a teacher to help her third-grade students learn the mathematics they needed to "know." On the one hand, Ms. Coleman wanted very much to help her students become responsible for their own learning and empower them to make their own competent judgments about the truth and usefulness of mathematical statements. At the same time, as much as Ms. Coleman wanted to explore mathematical ideas in depth and to give students time to work through their thinking, she had to keep a weather eye on the class's rate of progress. Ms. Coleman discussed in several interviews the pressure she felt to "cover content." She felt she had to make sure she "got in" certain topics and calculational procedures because they were things which the fourth grade teacher "expects when the children come." Also, at the time of this project, her district required all students to take the California Test of Basic Skills toward the end of third grade, and Ms. Coleman felt some responsibility to make sure her students would be ready for it. Because this test, like most standardized tests, places a premium on quick and accurate calculation, it measures and validates a type of mathematical competence than runs counter in many ways to the emphases of teaching for understanding. Ms. Coleman's need to prepare students for these experiences may have been one source of the "mixed messages" she conveyed to students and the "mixed bag" of what students came to understand as important mathematical knowledge.

Mathematical Knowledge

Different kinds of mathematical knowledge. In Ms. Coleman's class, mathematical knowing meant several different things. First, it meant knowing and understanding multiple methods or strategies for solving number problems. This was partly because that was how Ms. Coleman saw mathematics--as strategies for solving number problems--and partly because when students became the sources of the mathematical knowledge that was "taught," mathematical strategies and procedures emerged because these were what they had learned from parents and teachers or figured out for themselves. Multiple methods for solving number problems were discussed, and multiple methods were valued. At the same time, mathematical knowledge in Ms. Coleman's class also involved knowing number facts with speed and accuracy and being able to recall number facts quickly to solve problems. There was a press toward being able to solve mathematics problems with more abstract representations such as numbers and number sentences rather than using other representations such as pictures or objects. Some of what Ms. Coleman said and did in class, for example, her discouraging of the "sticks" because they "take so long," and her continued use of Mad Minutes, may have led some students, such as Chang, to retain a traditional view, of mathematical knowledge, or at least sent students somewhat mixed messages. Yet alongside a valuing of efficiency was also a valuing of reflection, as students often spent much class time reflecting on and trying to understand how a particular student was thinking about a problem solution. This led some students, such as Harold, to become self-reflective, often revising their solutions or strategies after considered thought and considerable discussion.

Different perspectives on how one comes to understand. Ms. Coleman often told her students to listen so they could explain to her what another student said because "when you understand, you're able to explain it in your own words." If a student couldn't "explain it in his own words," Ms. Coleman said that it showed her that maybe the student didn't understand. If a student couldn't explain what another student had said, Ms. Coleman would suggest that he or she was not paying attention. As she put it one day, "I have called on too many people who are not hearing what was being said." Somewhat like Ataia, Ms. Coleman seemed to assume that learning mainly involves listening--information is transmitted by the speaker and received by the listener. Understanding involves hearing what is being told to you by a speaker, and the message or the "knowledge" will be heard the same way by all the listeners. Often Ms. Coleman would remind the students, "I hope we're not too quick to say, 'I don't understand' because we weren't listening." Her implication was that if a student listened and heard, then that student would understand.

Ms. Coleman's perspective on how students would come to understand was revealed also when students "taught" each other during whole-class discussion. For example, one day Yan demonstrated how division can be repeated subtraction using the problem 40 divided 5. When students complained that they didn't understand, their teacher queried them, "When you don't understand something, what do you?" Students made confused noises in response so Ms. Coleman called on Yan to repeat what he said saying, "Yan is going to tell you." Later she referred to Yan as giving information. Again, Ms. Coleman's message was: The information is in the message, if the listener doesn't get it, then repeat it because the information is in the message and should be understood the same by each learner.

Although Ms. Coleman conveyed through her words and actions the idea that learning involves the transmission and receipt of mathematical "information" such as strategies for solving problems, her students constructed different understandings of this message. For most of the students we have discussed, coming to understand involved making meaning, transforming information, and inventing new ways of solving mathematics problems through classroom discourse and solving problems on one's own. Harold wanted to know why a strategy worked and be able justify his strategies and solutions to others before he was satisfied with his answers; and he asked the same of others, such as Calvin, when he queried him in their small group problem solving session. Selvaranee invented her own way of using "sticks" or tally marks to solve multiplication and division problems, and held to it as long as she needed it. Calvin continued to search for his own meaning in the school mathematics the class was doing but found little in the classroom discourse that related to his world. Yet when presented with contextualized problems, like the word problems in the interview or the story problems in the problem-solving session, Calvin understood and made sense of these problems in ways that the "best" math student in the class, Chang, could not accomplish. Chang, on the other hand, would readily agree with Ms. Coleman's perspective on understanding as involving transmission and receipt of information because that was how he had been taught mathematics by his parents.

Of all five students, Ataia seemed to be most uncertain about how one was supposed to go about "understanding" in this class. When she copied down information or listened to an explanation and took it literally, she was often called "wrong." She struggled to make sense of students' explanations for herself. Sometimes what Ataia heard literally did not correspond with her own sense of the mathematics. Often she did not resolve this discrepancy for herself, but instead she retained the ideas or solutions alongside one another. Continuing to retain multiple, often discrepant, ideas or multiple perspectives on the same problem is part of what led to our feeling that Ataia sometimes understood, and sometimes didn't. This led us to realize that different assessment situations--whole-class discourse, individual interviews by researchers, a small-group problem-solving interview, and student interviews by the teacher--all provide different perspectives or insights into students' mathematical understanding.

Different perspectives on mathematical understanding. A main goal of assessment, whether in research or in the classroom, is the same as one of Ms. Coleman's own goals for her new teaching--to find out what students really understand, "to know how they're thinking inside." Simply seeing that a student got an answer right told us, as researchers, little about how that student got it right--Did the student really understand the question or was he just get lucky in picking the correct procedure, or, if the test was multiple choice, did he just guess right?--or whether (like Chang) the student even knew what his answer meant. Simply seeing that a student got a problem wrong did not indicate why she got it wrong: she may have lacked any understanding of that problem, or she may simply have read the operation sign incorrectly, or have made some trifling error in computation, or maybe just needed a suggestion (like Ataia) that "borrowing" was an option she should consider here. Each of these different ways of getting problems "right" and "wrong" indicated a different type or level of understanding, each suggested very different learning needs, and none of these could be discovered by simply marking answers right or wrong and then recording the number right at the top.

As Ms. Coleman had hoped, class discussion revealed much about the thinking of at least her more vocal students. In this way, Ms. Coleman gained greater access to the knowledge and understandings of her students than could a traditional teacher who did not elicit such discourse from her students. The classroom discourse also gave us a window into students' mathematical understanding that we shared with Ms. Coleman.

Unlike Ms. Coleman, we had the additional advantage of windows into students' understanding through individual interviews and observing a small-group problem-solving session. We learned much about students' understanding by sitting down with students, watching them as they work to solve problems, and then asking them how they solved those problems, why they chose to proceed as they did, and whether they could think of any alternative way to approach the problem or to justify their answer. These student interviews typically took an hour or more. Teachers rarely have that much time to sit down with each of their students, watch them work on problems, and then talk about their work. During this unit Ms. Coleman took two days to interview students individually, but this gave her only about 10 minutes with each student, and she was only able to conclude that "everybody's kind of at a different place," which led her subsequently to attempt to "bring things to a common ground" during the last part of the unit. Watching and interviewing students while working in small groups also gave us access the thinking of quieter students often did not speak out in whole-class activities. From our small-group problem-solving sessions we learned that students' words and actions in a small group situation reveal a lot about how they understand learning--whether they see learning as essentially cooperative or competitive, and whether they have the necessary skills to learn and problem-solve in a less formal, peer-oriented situation.

We constructed our cases of five learners from information that we shared with the teacher, but also from information that we constructed and compiled later and which was not shared with the teacher. In this way we are offering our perspectives on individual students' understandings, and these were not necessarily evident, obvious, or even accessible to the teacher at the time. Indeed, we wonder how such in-depth perspectives on multiple students' understandings might become available to the classroom teacher in "real time." If such in-depth perspectives on students' understandings are deemed important, then teachers will need time and resources in order to be able to interview students, transcribe student comments, analyze student's thinking and work, and construct such knowledge of their students for themselves (see, for example, Rosaen & Roth, 1993). This kind of immediate "on-line" knowledge of their students' thinking and understanding is what teachers will need if they are ever to teach mathematics for understanding in the ways that reformers are advocating. In this way, these retrospective case studies have much to say prospectively.

Perspectives and Prospectives

From a teacher's perspective, this case study of an elementary teacher, her students, and her mathematics classroom offers one image of what might be possible as teachers transform their teaching in ways that encourage students' mathematical understanding. On the average, Ms. Coleman's students developed a greater repertoire of mathematical strategies and more flexible use of these strategies to solve number problems than similar students might in a typical, traditional third-grade mathematics class. Further, although these students came in with diverse backgrounds and understandings of mathematics, Ms. Coleman found that her nine-year old students really had a lot of mathematical knowledge, both as individuals and as a community of learners. They could make important substantive contributions to the classroom discourse. Together, her students could come up with strategies for solving mathematical problems without her "telling" them, and they could learn mathematics from each other. Throughout the six-week unit on multiplication and division, Ms. Coleman continued to be surprised, impressed, and delighted with the quality of her students' thinking, talking, and working during mathematics class. At the end of the year, Ms. Coleman planned never to return to her "old" way of teaching mathematics using CSMP, but rather she planned to expand her use of this discourse-based approach to teaching for understanding to her teaching of all subjects throughout the day.

From learners' perspectives, this case study offers multiple images of what it means to learn mathematics for understanding; mathematics, learning, and teaching were not the same for Harold, Chang, Calvin, Selvaranee, and Ataia even though these learners sat in the "same" mathematics classroom taught by Keisha Coleman. These children developed quite different understandings of what it meant to know mathematics and how they should go about learning mathematics. The learners "psychological realities" (Sarason, 1982), their understandings of the classroom discourse, their epistemological beliefs, and how they saw their roles as students, were all different, and those "realities" made all the difference to their actions and their understandings. Discourse, knowledge, and roles were interrelated such that transformations in any one of these opened up possibilities for transformations in the others.

From a researcher's perspective, as we watched and talked with Ms. Coleman and her students, we came to realize that teaching and learning for understanding are particularly challenging, precisely because there is no formula, no list of procedures to follow. Because such teaching and learning involve the collaborative creation of knowledge by teachers and students, with the aid of their own prior knowledge and experiences, texts, materials, and many other outside resources (including parents), and because classroom discourse is based, in part at least, on the students' own questions and developing understandings, no teacher's manual can lay out a plan for the year; there is no single, premarked path through, for example, the complexities of third-grade mathematics. For each teacher and group of students, "the going itself is the path" (Lewis, 1944, p. 68).

From a reformer's perspective, as we increasingly realized the complexity that evolved from the transformation of the classroom discourse, we came to increasingly respect Ms. Coleman and her students for their willingness to struggle with this sort of uncertainty, to break new paths for themselves. Ms. Coleman's role in this enterprise was pivotal. Much debate in the literature focuses on how much subject matter knowledge a teacher needs to teach for understanding. Certainly, Ms. Coleman might have found it easier to follow students' thinking and converse with students about important mathematical ideas had she felt more comfortable in her own mathematical knowledge. She might also have felt more confidence about her ability to handle certain students' ideas that seemed to come "from left field." Yet what is impressive here is the degree to which Ms. Coleman did enable students to construct important mathematical knowledge, and the way in which she herself was learning constantly. Ms. Coleman and her students together arrived at clear understandings of both multiplication and division, and they also developed a repertoire of strategies for solving problems in these areas. The students' fluent explanations and flexible use of these strategies during our final interviews show that, for most students, these strategies were not mindless, memorized procedures, but rather clearly understood ways of manipulating, representing, rearranging, and working with quantities, that is, of doing mathematics. Such flexible thinking and versatile use of a wide variety of strategies by children is a desired outcome of the reform because it is precisely that kind of mathematical thinking that characterizes expert mathematicians.[11]

As teacher educators, we found this a very hopeful result, because, frankly, if teaching mathematics for understanding can be done only by teachers who already have deep conceptual understandings of mathematics, it is not going to be done much at all. It would be especially difficult for elementary school teachers to develop a relatively complete conceptual base in *each* subject before starting to teach; no college program, however much it emphasizes the liberal arts, is long enough for that. Also, much is changing in our world and disciplinary knowledge is constantly being revised--the mathematics a teacher learns to teach today will need to be reconsidered when taught 20 years from now. Certainly, practicing teachers can enlarge and revise their subject matter and pedagogical knowledge through attending inservices and reading professional journals, yet most teachers, like the rest of us, have full lives outside of school--dealing with families, dual careers, and the tasks and stresses of daily living. In the current educational system, teachers have limited opportunities to engage in formal learning outside of the school day, when they also must continue to plan lessons, gather materials, and grade assignments. A teacher who can learn in her classroom, as she teaches, and *from her students* has the chance not only to enlarge her time and resources for learning, but also to model genuinely for her students the process and value of lifelong learning.

---

[11] For example, Dowker (1992) studied expert mathematicians and found that "the number of strategies used by different mathematicians for solving a single problem was startlingly large, especially in view of the fact that no attempt was made to elicit such variety. It became obvious that there is not always one single or even one optimum way of solving a problem" (p. 53).

From this experience as scholars, we deepened our commitment to what might be called "research for understanding," as represented by our work reported in this chapter. Such research does not start from a prescriptive set of teaching behaviors or learning outcomes, but rather begins by seeking to understand what learning and schooling look like to the teachers and students who are doing it, recognizing that "teachers" and "students" are not formal classes of objects for research, but individuals with their own diverse experiences, understandings, beliefs, and goals. These individuals live and work within their own communities, cultures, and contexts which also must be understood.[12] As researchers, we will probably always be seeking to understand, to build arguments and to persuade others of what we think we have discovered; we have, of course, done so in this very chapter. Yet we need to acknowledge that teachers and students are also, in their ways and according to their goals, seeking to understand, and that we have much to learn from them.

---

[12] For a similar perspective on research for understanding, see Sarason (1982).

References

Anghileri, J. (1985). Should we say times? Mathematics in School, 14(3), 24-26.

Anghileri, J. (1989). An investigation of young children's understanding of multiplication. Educational Studies in Mathematics, 20, 367-385.

Ball, D. L. (1990). Reflections and deflections of policy: The case of Carol Turner. Educational Evaluation and Policy Analysis, 12(3), 247-260.

Ball, D. L. (1993). With an eye on the mathematical horizon: Dilemmas of teaching elementary school mathematics, Elementary School Journal, 93(4), 373-397.

Ball, D. L., & Renquist, S. S. (1993). Collaboration as a context for joining teacher learning and learning about teaching. In D. K. Cohen, M. W. McLaughlin, & J. E. Talbert (Eds.), Teaching for understanding: Challenges for policy and practice (pp. 13-42), San Francisco: Jossey-Bass.

Ball, D. L., Kennedy, M., McDiarmid, G. W., & Schmidt, W. (1991). A study package for examining and tracking changes in teachers' knowledge (Technical Series 91-1). East Lansing, Michigan: National Center for Research on Teacher Learning, Michigan State University.

Boero, P., Ferrari, P. L., & Ferraro, E. (1989). Division problems: Meanings and procedures in the transition to a written algorithm. For the Learning of Mathematics 9(3), 17-25.

Brown, C., & Borko, H. (1992). Becoming a mathematics teacher. In D. A. Grouws (Ed.), Handbook of research on mathematics teaching and learning (pp. 209-239). New York: Macmillan.

California Department of Education. (1989). A question of thinking: A first look at students' performance on open-ended questions in mathematics. Sacramento, CA: Author.

Carpenter, T., & Fennema, E. (1990). Cognitively Guided Instruction: Multiplication and division. Madison: Wisconsin Center for Education Research, University of Wisconsin.

Cobb, P., Yackel, E., & Wood, T. (1989). Young children's emotional acts while doing mathematical problem solving. In D. B. McCleod & V. M. Adams (Eds.), Affect and mathematical problem solving: A new perspective (pp. 117-148). New York: Springer-Verlag.

Cobb, P., Wood, T., & Yackel, E. (1991). Analogies from the philosophy and sociology of science for understanding classroom life. Science Education, 75, 23-44.

Cobb, P., Wood, T., Yacket, E. & McNeal, B. (1992). Characteristics of classroom mathematics traditions. An interactional analysis. American Educational Research Journal, 29, 573-604.

Cohen, D. K. (1988). Teaching practice: Plus que ça change . . . In P. W. Jackson (Ed.), Contributing to educational change: Perspectives on research and practice (pp. 27-84). Berkeley, CA: McCutchan.

Crane, D. (1972). Invisible colleges: Diffusion of knowledge in scientific communities. Chicago: University of Chicago Press.

Davis, R. B., Maher, C. A., & Noddings, N. (1990). Constructivist views on the teaching and learning of mathematics. Reston, VA: National Council of Teachers of Mathematics.

Dowker, A. (1992). Computational estimation strategies of professional mathematicians. Journal for Research in Mathematics Education, 23(1), 45-55.

Fullan, M. (1985). Integrating theory and practice. In D. Hopkins & K. Reid (Eds.), Rethinking teacher education (pp. 195-212). Kent, England: Croom Helm.

Holt, J. (1967). How children learn. New York: Pitman Publishing.

Kennedy, M., & Barnes, H. (1993). Implications of cognitive science for teacher education. In J. Mangieri and C. C. Block (Ed.), Creating powerful thinking in teachers and students: Diverse perspectives. New York: Harcourt, Brace, Jovanovich.

Killion, K., & Steffe, L. (1989). Children's multiplication. Arithmetic Teacher. 37(1) 34-36.

Kouba, V. L. (1985). An investigation of Children's solution strategies for one-step, grouping and ratio forms of multiplication and division word problems. Unpublished doctoral dissertation, University of Wisconsin-Madison.

Kouba, V. L. (1989). Children's solution strategies for equivalent set multiplication and division word problems. Journal for Research in Mathematics Education 20(2), 147-158.

Kuhn, T. (1970). The structure of scientific revolutions. Chicago: University of Chicago Press.

Lakatos, I. (1976). Proofs and refutations. Cambridge: Cambridge University Press.

Lampert, M. (1986). Teaching multiplication. Journal of Mathematical Behavior 5, 241-280.

Lampert, M. (1987). Knowing, doing and teaching multiplication. Cognition and Instruction, 3, 305-342. (with commentary).

Lewis, C. S. (1944). Perelandra. New York: Macmillan.

May, W. T. (1989). Understanding and critical thinking in elementary art and music. (Elementary Subjects Center Series No. 8). East Lansing: Michigan State University, Center for the Learning and Teaching of Elementary Subjects.

Mid-continent Educational Research Laboratory. (1985). Comprehensive school mathematics program. Kansas City, MO: CEMREL.

National Assessment of Educational Progress (1983). The third national mathematics assessment: Results, trends, and issues. Denver, CO: Educational Commission of the States.

National Commission on Excellence in Education (1984). A nation at risk: The full account. Cambridge, MA: USA Research.

National Council of Teachers of Mathematics (1989). Curriculum and evaluation standards for school mathematics. Reston, VA: Author.

National Council of Teachers of Mathematics. (1991). Professional standards for teaching mathematics. Reston, VA: Author.

Peterson, P. L. (in press). Learning and teaching mathematical sciences: Implications for inservice education. In S. J. Fitzsimmons (Ed.), Pre-college teacher enhancement in science and mathematics: Status, issues, and problems. Washington, DC: National Science Foundation.

Peterson, P. L. (1990a). Cathy Swift: Doing more in the same amount of time. Educational Evaluation and Policy Analysis, 12, 261-280.

Peterson, P. L. (1990b). The California study of elementary mathematics. Educational Evaluation and Policy Analysis, 12, 241-246.

Peterson, P. L. (1992). Revising their thinking: Keisha Coleman and her third-grade mathematics class. In H. Marshall (Ed.), Redefining student learning: Roots of educational change (pp. 151-176). Norwood, NJ: Ablex.

Peterson, P. L., Fennema, E., Carpenter, T., & Loef, M. (1989). Teachers' pedagogical content beliefs in mathematics. Cognition and Instruction, 6(1), 1-40.

Peterson, P. L., & Knapp, N. F. (1993). Inventing and reinventing ideas: Constructivist teaching and learning in mathematics. In G. Cawelti (Ed.), Challenges and achievements of American education: the 1993 ASCD yearbook. Alexandrai, VA: Association for Supervision and Curriculum Development.

Peterson, P. L., Putnam, R. T., Vredevoogd, J., & Reineke, J. (1992). Profiles of practice: Elementary school teachers' views of their mathematics teaching. In W. G. Secada (Ed.), The reform of school mathematics in the United States, [special issue] International Journal of Educational Research, 17, 471-488.

Remillard, J. (1990). Is there an alternative?: An analysis of commonly used and distinctive mathematics curricula (Elementary Subjects Center Series No. 31). East Lansing: Michigan State University, Center for the Learning and Teaching of Elementary Subjects.

Resnick, L. B. (1989). Developing mathematical knowledge. American Psychologist, 44(2), 162-169.

Rosaen, C. L., & Roth, K. J. (1993). Similarities and contrasts bewtween writing during a writers' workshop and writing in science: Examining the teacher's role (Elementary Subjects Center Series No. 94). East Lansing, MI: Michigan State University, Center for the Learning and Teaching of Elementary Subjects.

Roth, K. J. (1989). Conceptual understanding and higher level thinking in the elementary science curriculum: Three perspectives (Elementary Subjects Center Series No. 12). East Lansing: Michigan State University, Center for the Learning and Teaching of Elementary Subjects.

Sarason, S. B. (1982). The culture of the school and the problem of change. (2nd ed.). Boston: Allyn & Bacon.

Schifter, D., & Fosnot, C. T. (1993). Reconstructing mathematics education: Stories of teachers meeting the challenge of reform. New York: Teachers College Press.

Schoenfeld, A. (1992). Learning to think mathematically: Problem solving, metacognition, and sense making in mathematics. In D. A. Grouws (Ed.), Handbook of research on mathematics teaching and learning (pp. 334-370). New York: Macmillan.

Steffe, L. P., & Von Glasersfeld, E. (1985). Child generated multiplying and dividing algorithms: A teaching experiment. Washington, DC: National Science Foundation.

Vernaud, G. (1983). Multiplicative structures. In R. Lesh & M. Landau (Eds.), Acquisition of mathematics concepts and processes (pp. 128-175). New York: Academic Press.

Wilkinson, L. C., & Marrett, C. (1985). Gender influences in classroom interaction. New York: Academic Press.

Wilson, S. M., & Wineburg, S. S. (in press). Wrinkles in time and place: Using performance assessments to understand the knowledge of history teachers. American Educational Research Journal.

APPENDIX A

METHOD

This report is based on extensive data gathered during the academic year 1989-90. Data were gathered through interviews and observations and were subjected to both qualitative and quantitative analyses.

Interviews With the Teacher

Between November 1989 and October 1990 Penelope Peterson conducted 17 interviews with the Ms. Coleman. All but two interviews were conducted following mathematics lessons that Ms. Coleman has just taught and Peterson had just observed. Peterson began each interview with questions adapted from Post-Observation Questionnaires developed by the National Center for Research on Teacher Learning (Ball, Kennedy, McDiarmid, & Schmidt, 1991) and the Education Policy and Practice Study group (Peterson, 1990b) at Michigan State University. As each interview progressed, Peterson relied not so much on the structured interview format as on her own knowledge and experience gained from interviewing Ms. Coleman over time and from interviewing other elementary teachers. She asked Ms. Coleman questions that probed in depth the teacher's thinking about mathematics teaching and learning in her classroom and about her students.

All 17 interviews were audiotaped and transcribed using a word processor, and thus were available for analysis both on paper and on the computer. In outlining the direction of Ms. Coleman's intended teaching reforms, as part of the context for the student cases, we paid particular attention to the interview on 11/17/89, in which she talked about how observing her colleague's, Deborah Ball's, teaching had inspired her to make certain changes in her own teaching, and to one on 1/11/90, in which she discussed the goals of these changes as they were unfolding.

Interviews With Individual Students

Procedure

We interviewed each student in Ms. Coleman's class in October, March, and June of the project year. The March interview, shorter and less general than the other two, was primarily concerned with students' addition and subtraction with regrouping, so this chapter, which focuses primarily on the multiplication and division unit which Ms. Coleman taught in April, does not include data from it. Both the October and June interviews included six addition and subtraction computation problems and several word problems as well as questions about the nature and uses of mathematics and how the students perceived themselves and others in mathematics. The June interview also included questions about the students' perceptions of their mathematics class during the project year and 12 computational problems in multiplication and division which we asked students to solve and then explain to us. We encouraged students to take as long as they needed and to solve problems in any way they chose; several manipulative and other aids were available to them in addition to pencil and paper, including base-10 blocks, bean sticks, a number line, and a chart of basic multiplication facts.[13]

One student moved after the first interview, while two students with limited English proficiency seemed unable to understand or reply to most of the interview questions, so these students were not included in the final analysis, leaving 15 students in the data set. All interviews were audiotaped, transcribed, and verified for accuracy. Student productions such as drawings and written calculations were saved, each of us took notes on the behaviors of the students we interviewed, and these data were integrated into the interview transcripts. Most of the June interviews were videotaped as well; viewing the videotapes allowed us to verify and amplify these notes. When the transcripts were in final form, the data were analyzed in several different ways.

---

[13] Selected questions and problems from the October and June interview protocols appear in Appendix B.

Qualitative Analysis of Student Interview Data

Nancy Knapp used Hyperqual, a yypercard-based program developed by Dr. Raymond Padilla of Arizona State University, as the main tool for qualitative analysis. We formulated 13 areas as conceptual organizers, which took the form of questions, some correlating to specific interview questions and others for which evidence was more generally distributed throughout each interview. Six of these conceptual questions were addressed by material from both October and June interviews, while the other seven, because they dealt specifically with multiplication and division, were addressed only during the June interviews. These conceptual questions were as follows:

Oct. & June
1. What is "being good" at math, and how do people get that way?
2. Is math enjoyable? Is it hard or easy?
3. What is Mad Minutes like?
4. What is the student's self-concept in math?
5. What is it like when someone disagrees with you in class?
6. How can you tell if an answer you get is right?

June only
7. Was this year's math class different from last year's? How?
8. What is a "method, and which ones are remembered/used?
9. What is multiplication?
10. What is division?
11. How are they related and different?
12. How does the student solve, prove, and explain the multiplication problems on the interview?
13. How does the student solve, prove, and explain the division problems on the interview?

Later, we developed a particular interest in "Chang's method," and we formulated an additional question:

14. What is "Chang's method," how did it develop, and how can it be used?

Excerpts were chosen from each student's interviews that provided evidence of that student's beliefs and ideas in each area. For example, the following excerpt was selected as a part of the evidence about Ataia's self-concept in mathematics in October:[14]

---

[14] For ease in distinguishing, the students' remarks were put in quotations, while the interviewer's were left without. The number at the beginning of the excerpt refers to the page number of the printed transcript where this passage can be found. In

-2-What subject did you think you did the best in last year?
"Math."
Math, you think.
"Yeah."
Why did you think you did the best in math? (5-second pause) Why did you think that you did the best?
"Because on my math I always came home with stars and stuff."
Because you always came home with stars and stuff.
"Uh-huh."
Do you like math?
"Not that much."

Excerpts giving evidence of how a student solved a problem often contained both dialogue and references to the student's written work, drawings, or actions with manipulatives. The following example shows how Tessa solved the problem 60 ÷ 10 on the June interview.

-15-(She writes 6 right away)
"And 60 divided by 10, well, it's like, um, I have. This is 60 . . . (she holds up 6 fingers.)"
Okay. There's 60. There's six fingers there. How do I know it's 60?
"Just . . ."
Just what?
". . . pretend this is 10 (one finger), this is 20 (two fingers) . . . "
Oh, each finger's being a 10. Okay.
"Okay."
Go with it.
"And you just count 'em . . .
Oh.
". . . 1, 2, 3, 4, 5, 6 (counting fingers)."
Uh huh.
"And then that's how I got that answer."

All passages related to a particular area from each student's interviews were placed on a single "card" in Hyperqual. The 14 "cards" for each student were then sorted and resorted on the computer to compile evidence from each particular area across students and also to juxtapose responses from related areas for each student. Finally two summaries of evidence were constructed for each student: one for areas 1-7, covering the student's general orientation to mathematics and this class; and one for areas 9-13, covering specific comments, definitions, and problem-solving strategies related to

---

this way, we were always able to trace a passage back to its original context, if we had any questions about its interpretation.

multiplication and division. These summary sheets, whose contents could be traced back to individual cards, and, if necessary, further back to pages in the interview transcriptions, were primary sources in preparing the case studies that form the body of this report.

Quantitative Analysis of Student Interview Data

These summary data was also categorized and eventually coded and arrayed on three spreadsheets: one detailing students' responses on all questions in common between the two interviews; one summarizing primarily affective and epistemological data for the students, such as whom they named as "good" at math and why, and noting changes from the beginning to the end of the year; and one identifying the various strategies students used to solve or explain each multiplication and division computation problem on the final interview, including who used which strategies and whether they were successful or unsuccessful.[15] These spreadsheets provided a good overall, but highly capsulated, view of the student interview data as well as enabling us to do some basic quantitative analysis, such as figuring classroom means for numbers of strategies used or correlations between strategies attempted and problems solved. Much of the information from these spreadsheets is not included in this report, which focuses on cases of five individual students, but some appears in the section entitled "Flexible Strategy Use."

Small Group Problem Solving Sessions

In June, one to two weeks after school was out, 12 of the 15 children in our data set consented to participate in small group problem solving sessions. Four groups were formed of three children each. These sessions took approximately two hours, and were structured around several group tasks. Not all groups reached the last two tasks, so data from only the first task were used. The first task related to the following set of word problems:

---

[15] The categories of strategies on this last spreadsheet were derived both from constructivist literature on elementary work in multiplication and division (e.g., Anghileri, 1985, 1989; Boero, Ferrari, & Ferraro, 1989; Carpenter & Fennema, 1990; Killion & Steffe, 1989; Kouba, 1985, 1989; Lampert, 1986, 1987; Steffe & Von Glasersfeld, 1985; Vernaud, 1983) and from the students' own variety of strategies. Explanations of the 15 categories used, plus examples demonstrating how responses were coded, appear in Appendix C.

> A. Marla has a job after school. Last week she worked two hours and earned \$10.50. How much did she earn per hour?
>
> B. This week Marla worked two hours and earned \$10.50 per hour. How much did she earn this week?
>
> C. Marla worked two jobs. She earned \$5.25 on the first job and \$10.50 on the second job. How much did she earn at both jobs together?

Each student was provided with a copy of these problems. Penelope Peterson conducted the sessions and read the problems aloud. She asked each group to come to an agreement about which two of the above problems were most alike, and which two were most different, and why. Although the students were not asked to solve the problems, they all invariably began their work by doing so. On the table around which the group sat were available a number of aids, including beansticks, play money, papers, pencils, and markers. Large sheets of paper were hung on the wall to simulate a blackboard. The entire group process was videotaped and audiotaped, Peterson took fieldnotes, and all student written work and drawings were saved. The tapes were later transcribed, and Nancy Knapp verified and augmented these transcripts through viewing the videotapes and incorporating data from fieldnotes and student.

Classroom Observations

This report focuses especially on the unit Ms. Coleman taught on multiplication and division. This unit covered 20 days of mathematics instruction in April and early May, 18 days of class discussion, and 2 days during which Ms. Coleman interviewed individual students about their solutions and strategies on a worksheet she had assigned as homework. We observed, videotaped, and audiotaped all 20 days of this unit. Nancy Knapp reviewed each videotape, taking detailed notes of the sequence of events and keeping a count of which students participated in each class session and who made significant contributions to the discussions. Using these notes along with the audiotape transcriptions, she then wrote two-four page summaries of each day's class, paying particular attention to various "methods" as they were proposed, refined and used, and to discourse-related events, such as statements from Ms. Coleman outlining classroom norms or occasions when students' contributions were or were not accepted in the class. In these summaries, each event was numbered by the counter on the tape player, so that if necessary one could go back and view again what had occurred and the sequence of events leading up to and following it. These summaries form the other major part of the data base (the first being the individual and group student interviews) from which the case studies in this report were constructed.

Other Data Sources

We obtained copies of most of the students' math notebooks. These were less helpful than we had hoped, since most students worked in them only sporadically, and in students did not seem to use notebooks for general note-taking, but only for recording what Ms. Coleman told them to write down. However, in several cases, information in the notebooks helped to clarify the understanding behind a student's recorded statement in class, or to indicate a students' progressive refinement of a method of solution over a period of time. We also were given copies of most of the students' score reports from the California Test of Basic Skills (excluding those who had moved over the summer), but again found that they were less than helpful, since the CTBS test is based on very different assumptions and goals from those which formed a foundation for what Ms. Coleman was trying to do in her mathematics class. However, we did find some confirmation for one students' unexpected expertise in math in her equally unexpected high score on the CTBS test (see "Selvaranee's story" below). In addition, Ms. Coleman provided us with copies of each student's grades on the Mad Minutes quizzes given in class throughout the year.

APPENDIX B

SELECTED QUESTIONS FROM OCTOBER AND JUNE STUDENT INTERVIEW PROTOCOLS

Some of the questions that we asked students in October and again in June were adapted from interview protocols used by Magdalene Lampert and Deborah Ball in the National Science Foundation project in which they have been documenting the learning in their elementary mathematics classrooms. The questions below are a selected subset of the questions that we asked students. We present these questions because in our case analyses of students in this chapter we focus on students' responses to these questions. Although interview protocols were scripted, we did not always use the exact scripted words in asking questions, if we thought that slightly different wording would fit the flow of the conversation better. Also, we frequently asked students probing or clarifying questions to gain a better understanding of what students meant by their answers. We present the questions below according to the major issues or ideas that they were intended to explore.

ACADEMIC ATTITUDES AND EXPERIENCES:

October only:

1. Last year, what was your favorite subject? Why was that your favorite?
2. What was your least favorite subject? Why?
3. What subject do you think you did best in last year?

June only:

1. This year, have you had a favorite subject? Why was that your favorite?

EXPERIENCES WITH AND ATTITUDES TOWARD MATH

October and June:

1. Do you like math? Why or why not? (*skip if answered in previous section*)

October only:

2. What is the most fun thing for you in math? What makes it fun?

3. What is the hardest thing you have learned so far in math? What made it hard?

June only:

2. What is the part of math you like the best?

3. Do you think math is hard? What is the hardest thing you have learned so far in math? What made it hard?

4. Was your math class this year different from your math class last year? (*Probe for "different talking" if necessary.*)

IMAGES OF COMPETENCY IN MATHEMATICS (in October and June)

1. Do you know someone who is really good at math?

2. What does _____ do to make you think he/she is really good at math?

3. How do you think _____ got to be so good at math?

SOCIAL IMAGES OF MATHEMATICS/MATHEMATICAL EPISTEMOLOGY

October and June:

1. When you were in math class and the teacher would give everyone a problem to work on, and different people came up with different solutions to the problem, did people ever challenge (JUNE--"disagree with") one another's solutions?

2. Did anyone ever challenge (JUNE--"disagree with") you?

3. (*If yes*) How did that make you feel? (*If no*) How do you suppose you would feel if that happened?

4. When you are doing a math problem, how can you tell if you got the answer right?

June only:

5. In math class, I hear you all talking about different children's methods a lot. Can you explain to me what a method is? Could you tell me about a method that a student came up with in class?

PLACE VALUE, ADDITION AND SUBTRACTION--(October and June)

1. (*Show child the blue cards with each of the following problems on them and read them to him/her. Child may use whatever strategy he/she chooses: write or do these in his/her head.*)
   a. What is 25 + 10?
   b. What number is ten less than 40?

(*If child does these on paper, ask*; Could you figure these out another way, without paper and pencil?)

c. What is 326 - 100?
d. What is 326-99?

2. Could you do these problems? (*Hand child sheet with these problems--make sure manipulatives and pencil are available*)

$$\begin{array}{r}46\\ \underline{+28}\end{array} \qquad\qquad \begin{array}{r}50\\ \underline{-14}\end{array}$$

(*for each one*) Could you explain to me how you did this?

MULTIPLICATION AND DIVISION

Conceptions/definitions--June only

1. Can you tell me what multiplication is?

2. Can you tell me what division is?

(*In each case, if child gives an example,* ask "Why is that _____?" *Try to encourage child to come up with a definition that does not depend on the example)*

3. How would you explain to a second grader how these problems are the same or different? (*Show card with the following problems.*)

12 x 2   12 ÷ 2

Word problems--October and June

1. (*the MARKER problem*) There are 26 students in a class. the teacher wants each group of four students to have one set of markers to share. How many packs of markers should the teacher buy?

2. Suppose you were having a little picnic lunch with 4 kids and you made 8 sandwiches. How many would you serve to each kid if you wanted to give them the same amount? Could you show me how you figured that out? (*If necessary, clarify that the total kids includes him/herself*)

3. What if you only made 6 sandwiches? How much would you serve to each kid? Could you show me how you figured that out?

October only:

4. Suppose you wanted to be able to give everyone 3 sandwiches. How many would you need to make? Could you show me how you figured that out?

June only:

4. How about if you really made a mistake--you invited 8 kids and only had 4 sandwiches? How much would you serve to each kid, if you wanted each kid to have the same amount?

Calculation problems - June only

(Give child sheet with the following problems. Make sure number line, times table, base-10 blocks, beansticks, and pencil are all readily available and that child understands he/she may solve in any way he/she prefers and take as long as he/she wants--note observable process)

1. $\begin{array}{r} 6 \\ \underline{\times 9} \end{array}$

2. $\begin{array}{r} 10 \\ \underline{\times 6} \end{array}$

3. $60 \div 10 =$

4. $\begin{array}{r} 12 \\ \underline{\times 4} \end{array}$

5. $56 \div 8 =$

6. $600 \div 10 =$

8. $\begin{array}{r} 100 \\ \underline{\times \ \ 8} \end{array}$

9. $23\overline{)230}$

10. $11\overline{)132}$

11. $8\overline{)4}$

(When child has done all he/she can, for each problem ask, How did you get your answer?)

Appendix C

CATEGORIES USED TO CODE STUDENTS' MULTIPLICATION AND DIVISION STRATEGIES

Strategies adopted by Keisha's students to solve the 12 numerical multiplication and division problems on the June interview were coded and grouped as follows:

Times Table: Problems are solved by direct reference to the chart of basic multiplication facts provided.

> EXAMPLE of Bert explaining 6 x 9:[16] "I went to the 6 column (demonstrating on times table) so go to the 9, I run down six times, 1, 2, 3, 4, 5, 6. See if that is number 6, so I run my finger down to 54."

Counting strategies:

Direct representation of each unit: A finger, a tally, a unit block or some other object is used to represent each unit in numbers given in the problem. These units are operated upon by counting the total (multiplication) or by dividing into groups somehow (division).

> EXAMPLE of Melissa doing 6 x 9: (She gets out ten base-10 "longs") "1, 2, 3, 4, 5, 6, 7, 8, 9, 10 . . . oops! (puts back one long) Okay, so I'll count these like this, 1, 2, 3, 4, 5, 6, 7, . . . 54. (counts each square of first 6 rows on blocks horizontally, checking twice to see if she had counted six rows only)."

Number line: Problems are solved by using spaces or groups of spaces demarcated on the number line

> EXAMPLE of Chang explaining 12 ÷ 2: "If you counted 2, 4, 6, 8, 10, 12, that's 1, 2, 3, 4, 5, 6. (He is counting by twos on the number line from 0 to 12, spanning each two spaces with his fingers.)"

Representation of each set: Each group is represented by one block, finger, tally or other object as the student counts up to obtain a total (multiplication) or notes the number of groups it takes to count up to a predetermined total (division).

> EXAMPLE of Frankie explaining 6 x 10: "10, 20, 30, 40, 50, 60 . . . 6. (He is counting by tens on his fingers, one for each ten.)"

---

[16] Note that for ease of distinction, again the students' comments are in quotations, while any interviewers' remarks are not.

Counting by sets mentally: The same as the above strategy, only no physical representation is used for each set.

Addition strategies:

Addition fact: A memorized addition fact is used to solve a multiplication or division problem.

> EXAMPLE of Josephia explaining how she got 4 for 4 + 8:
> (She has written two 4s, and then crossed one off)
> Ok, um, let's look at this, what did you get for an answer there?
> "Four."
> Just 4, because I'm seeing two 4s, are there supposed to be two there or not?
> "No, I crossed that one out. (She crosses out first 4 more heavily.)"
> Oh, that one's Ok, so the answer is just 4, now how did you get that answer?
> "Because if it's two 4s, I added 4 two times."
> Hmm
> "And I came up with 8."
> (Note that this strategy was not successful for her on this problem.)

Column addition: A column of one factor repeated is added to solve a multiplication problem.

> EXAMPLE of Harold explaining 8 x 100: "A hundred eight times is eight hundred. You can add up the zeroes. (He writes down a column of eight 100's and adds them on his paper to get 800.)"

Grouped addition: Problems are solved by adding factors in groups, rather than singly as in repeated or column addition.

> EXAMPLE of Ali explaining 4 x 12: "Well, in here I group numbers, like 12 plus 12, that's 24, and I would add 24 plus 24, and it equaled, um, to 48."

Place value strategies:

Chang's add-a-zero method: A method detailed by Chang in class of muitiplying a number by 10 by writing a 0 after it.[17]

> EXAMPLE of Selvaranee discussing 6 x 10:
> (She does the problem in her head)
> How'd you know that one?
> "Because I used Ben's method that I told you, that anything you divide (sic) by 10 just carry the 0 and put it here (very lightly writes 0 after the factor 6)".

---

[17] For further information on this strategy and the following one, see "Chang's story" in the body of this report.

Some students extrapolated this method to cover multiplying by 100.

> EXAMPLE of Marta explaining how she did 8 x 100:
> "I did it almost like this one (6 x 10) except this was 10 [and this] was a hundred, so then I put 800 because I think it works like that."
> How does it work? Like what?
> "Like that, like Ben's method, with a hundred."
> How does it work with a hundred then?
> "Um, you just add 8, I mean you add um, the 8, two 0's."

Chang's cross-off-a-zero method: A related method, also proposed by Ben, for dividing a number ending in 0 by 10 by crossing out the 0.

> EXAMPLE of Selvaranee justifying why 600 + 10 = 60:
> "I knew it because, if it's times, you, like it's 60 x 10, [you] just add the 0."
> Mm-hmm.
> "But if it's divide, you just take off the 0."
> Oh, I see, so if you, if you multiply by 10, you add a 0...
> "So, I, so, yeah."
> ...and if you divide by 10, you, what do you do when you divide by 10, just?
> "Erase the 0. "
> Oh, okay.
> "Take away the zero."
> So that's what you did?
> "Yeah, I did from the 600."

Other place-value related methods: These solutions seemed to be based on general place-value related thinking.

> EXAMPLE of Yan explaining why 600 + 10 = 60: "[It's] like I'm just doing like 60 divided by 10 = 6. (He points up to that problem.)"
> Ok.
> "And now it's 600, so I just added 0, because there is two zeros in that."
> Because there is two zeros in 600?
> "Mm-hmm (affirmative)".

Algorithmic strategies:

Known fact: The student simply remembers the answer to a problem.

> EXAMPLE of Calvin describing how he got 56 + 8:
> How'd you know that was seven?
> "I just predicted."
> Is it just in your head, you mean?
> "Yeah, I just, it just came up. It just came up."

Derived fact: The student finds a solution by figuring from a related, remembered number fact.

> EXAMPLE of Arnie solving 6 x 9: "I know that 8 times 6 equals 48, . . . so six, six more is, um, 54."

Division via multiplication: A division problem is solved by recasting it as a multiplication problem with a "missing factor," or justified by reference to the related multiplication problem.

> EXAMPLE of Chang solving 56 ÷ 8:
> "Mmm, 56, 56 divided by 8, like 7 x 8 = 56."
> Well, what did you say in your head? I mean, how did you know that was a 7?
> "I thought I could figure out what times 8 equals 56."

Conventional algorithm: The student uses the conventional multi-digit multiplication or long division algorithm to solve a problem.

Although for the sake of clarity, most of the above examples are of successful strategy use, each of these strategies was also attempted by some students one some problems with unsuccessful results. The strategy was still coded the same way, but the lack of success was noted. Also, on many occasions a student demonstrated more than one strategy for solving or proving a problem. Each strategy explicated was coded for that problem, along with its effectiveness.

> EXAMPLE of Bert doing 600 ÷ 10:
> (He gets out 6 flats, counts ten rows of 10, on one flat) "1,2,3,4,5,6,7,8,9,10 (then counts up the rest by tens) 20-30-40-50-60".
> How did you do that one (600 ÷ 10)?
> "I did this one fast."
> I know.
> "Any number divided by 10, you just minus a 0."
> You just . . .
> "You take away a 0."
> You do?
> "Like if this would be 60 times..."
> So you do 600 minus 0 . . .
> "If this was 60 x 10 it would equal 600, but since it's 600 divided by 10, it equals 60. I just took off the 0. (He crosses off final 0 of 600.)"
> You just took off the zero.
> "Off of the 600."
> OK, is that how you thought of doing it the first time?
> "Yes."
> Cause I saw you get out a few of those (B10 flats), or were you just sort of checking it?
> "I was just checking it."

This response would be coded as three different strategies: representation of each set, Chang's cross-off-a-zero method, and division via multiplication.

Occasions on which a student used an unknown strategy, performed the wrong operation (such as adding 4 + 12 instead of multiplying 4 x 12), or would not attempt the problem were also coded separately.